International Journal of Non-Linear Mechanics 67 (2014) 153–163

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

On approximating the effective bandwidth of bi-stable energy harvesters

Meghashyam Panyam *, Ravindra Masana, Mohammed F. Daqaq

Nonlinear Vibrations and Energy Harvesting Lab.(NOVEHL), Department of Mechanical Engineering, Clemson University, Clemson, SC 29634, United States

ARTICLE INFO

Article history:
Received 14 April 2014
Received in revised form
21 July 2014
Accepted 3 September 2014
Available online 16 September 2014

Keywords:
Vibratory energy harvesting
Non-linear dynamics
Bi-stable oscillators
Bifurcations
Effective bandwidth

ABSTRACT

This paper aims to establish an analytical framework to define the effective bandwidth of bi-stable vibratory energy harvesters possessing a symmetric quartic potential function. To achieve this goal, the method of multiple scales is utilized to construct analytical solutions describing the amplitude and stability of the intra- and inter-well dynamics of the harvester. Using these solutions, critical bifurcations in the parameters' space are identified and used to define an effective frequency bandwidth of the harvester. The influence of three critical design parameters, namely the time constant ratio (ratio between the time constant of the harvesting circuit and the period of the mechanical system), the electromechanical coupling, and the shape of the potential function, on the effective frequency bandwidth is analyzed. It is shown that, while the time constant ratio has very little influence on the effective bandwidth of the harvester, increasing the electromechanical coupling and/or designing the potential function with deeper potential wells serve to shrink the effective bandwidth for a given level of excitation. In general, it is also observed that narrowing of the effective bandwidth is accompanied by an increase in the electric output further highlighting the competing nature of these two desired objectives.

1. Introduction

Deliberate introduction of stiffness non-linearities into the design of vibratory energy harvesters remains one of the most researched topics in the energy harvesting community [1–17]. Driven by the ability of the non-linearity to extend the coupling between the harvester's response and the excitation to a wider range of frequencies, many research studies have demonstrated that non-linearities can be used to decrease sensitivity to parameter uncertainties and to enhance performance under random and non-stationary excitations commonly encountered in realistic environments.

Nonetheless, the complexity of the response behavior of non-linear vibratory energy harvesters (VEHs) as compared to their linear counterparts introduces additional challenges that complicates the full characterization of their response, thereby reducing our ability to reap their full benefits. Non-linear harvesters have been shown to exhibit different behaviors that are not seen in linear systems including sub-harmonic, super-harmonic, quasi-periodic, aperiodic, and chaotic responses. They can also undergo different bifurcations in the parameter space which yield sudden jumps in the response amplitude and/or switching in its period (doubling/halving).

Such complex responses are most commonly encountered in VEHs that incorporate a bi-stable potential energy function. In these devices, the non-linearity produces a potential energy function with two minima (stable equilibria) separated by a potential barrier (an unstable saddle) [4–8]. Due to the presence of the two potential wells, the performance of the bi-stable VEH is dependent on the excitation's level; if the excitation level is too small to activate the inter-well oscillations, the dynamics remain confined to one potential well producing small-amplitude responses that are not particularly favorable for energy harvesting. When the excitation is large enough to allow the desired large-amplitude inter-well oscillations, the harvester can perform complex non-unique dynamic responses including inter- and intra-well chaos as well as periodic responses at the excitation frequency or fraction integers of it [18].

Due to these complex responses, many researchers have pointed to the difficulty of assessing the performance of bi-stable energy harvesters and to achieve an optimal design [5,19]. Firstly, without prior knowledge of the intensity of the excitation source, the harvester potential can be designed to be too shallow for the bi-stability to be useful or too deep for the dynamic trajectories to escape a single potential well [9]. Secondly, even when the potential function is properly designed for the excitation

* Corresponding author.
E-mail address: mpanyam@g.clemson.edu (M. Panyam).

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.09.002

level, the large-orbit branch of inter-well periodic motion is not always unique and can be accompanied by a chaotic attractor and small branches of undesired intra-well oscillations [7,10,17]. In fact, it has been demonstrated through numerical simulations that the bandwidth of frequencies where the desirable large-orbit branch of periodic solutions is unique has a complex dependence on the design parameters including the potential shape, the electromechanical coupling, the effective damping, and most importantly the level of excitation. This complex dependence cannot be resolved by depending solely on numerical simulations or sets of experimental data. Analytical and semi-analytical approaches,[1] similar to those recently proposed by [24–27], are becoming more critical to delineate this dependence, and to propose techniques to possibly expand this effective bandwidth.

In this paper, we implement perturbation methods, namely the method of multiple scales, to construct equations that govern the slow modulation of the response amplitude and frequency. These equations are then used to investigate the steady-state amplitude and stability of the periodic output of the harvester as it performs either intra- or inter-well oscillations. Based on these responses, key bifurcations in the excitation's amplitude–frequency parameter space are identified and used to construct several maps that define the effective bandwidth of the harvester for several design parameters.

The paper is organized as follows. Section 2 provides a brief description of the mathematical model governing the dynamics of a typical bi-stable VEH. Section 3 studies the response behavior of the harvester to harmonic excitations with fixed-frequency characteristics. Section 4 presents the approximate analytical solutions developed to identify critical bifurcations in the parameter's space and the influence of the electric parameters on them. Section 5 defines an effective bandwidth for bi-stable VEHs and studies the influence of different potential shapes on it. Finally, Section 6 provides important observations and conclusions.

2. Mathematical model

Several lumped and distributed-parameter models have been developed to describe the dynamics of VEHs [28,29]. For the most part, these models are device specific and not very well suited to develop a qualitative understanding of the response behavior. In order to gain the insights necessary for a more general understanding, we consider a canonical model consisting of a mechanical oscillator coupled to an electric circuit through an electromechanical coupling mechanism. The circuit can be a first-order RL circuit representing an inductive transduction mechanism (e.g., electromagnetic harvester) or a first order RC circuit representing a capacitive transduction mechanism (e.g., piezoelectric harvester). In both cases, the non-dimensionalized equations of motion can be written in the following general form:

$$\ddot{x}+2\zeta\dot{x}+(1-r)x+\delta x^3+\kappa^2 y=\mathcal{F}\cos(\Omega t), \tag{1a}$$

$$\dot{y}+\alpha y=\dot{x}, \tag{1b}$$

where the overdot represents a derivative with respect to non-dimensional time, t. The variable x represents the displacement of the oscillator mass, ζ represents the mechanical damping ratio, $(1-r)$ is a linear stiffness coefficient where r is introduced to permit variations in the linear stiffness around its non-dimensionalized value of one, δ is the coefficient of cubic non-linearity, κ^2 is a linear dimensionless electromechanical coupling coefficient, y is the electric quantity representing the induced voltage in capacitive harvesters and the induced current in inductive ones, α is the ratio of the mechanical and electrical time constants of the harvester. The time constant of an inductive circuit is L/R and for a capacitive circuit is RC. Finally, $\mathcal{F}\cos(\Omega t)$ represents the external excitation term with $\mathcal{F}$ being the amplitude, and Ω the frequency of excitation.

Since this study is focused on the analysis of bi-stable VEHs, we limit our attention to the case when $r>1$, and $\delta>0$. In such a scenario, as shown in Fig. 1, the quartic potential energy function is bi-stable with the following three extrema:

$$x_s=0, \quad x_s=\pm\sqrt{\frac{(r-1)}{\delta}}, \tag{2}$$

where the maximum occurring at $x_s=0$, represents an unstable saddle, while the global minima, $x_s=\pm\sqrt{(r-1)/\delta}$, represent stable equilibrium solutions (nodes).

3. Response to harmonic fixed-frequency excitations

Bi-stable VEHs are capable of producing large amplitude responses over certain frequency ranges under harmonic excitations [5]. These responses occur when the excitation amplitude is large enough to permit the dynamic trajectories to escape the basin of attraction of a single stable node allowing the harvester to perform inter-well motions. Unfortunately, these desired motions cannot be uniquely realized over a large frequency bandwidth and are often accompanied with other, less desirable, small amplitude intra-well responses. To further illustrate this issue, Eqs. (1a) and (1b) are numerically integrated to construct a bifurcation diagram of the frequency response for different excitation amplitudes as depicted in Fig. 2.

When the normalized excitation amplitude is relatively small, $\mathcal{F}=0.045$, as shown in Fig. 2(a), the dynamic trajectories remain confined to a single potential well because the excitation is not large enough for them to overcome the potential barrier (the saddle) and escape from the well. As such, the harvester cannot perform the large-amplitude inter-well oscillations desired for energy harvesting. The frequency–response curve appears to be of the softening nature with the large amplitude resonant oscillations, B_r, occurring at frequencies smaller than the resonance frequency. The response curve undergoes two bifurcations: The first occurs as the frequency is decreased and the resonant branch, B_r, loses stability through a cyclic-fold bifurcation, *cfB*, giving way to the smaller non-resonant branch, B_n. The second occurs when the frequency is increased and the branch, B_n, undergoes another cyclic-fold bifurcation, *cfA*, giving way to the resonant branch, B_r.

As shown in Fig. 2(b), when the excitation is increased to $\mathcal{F}=0.11$, another large-amplitude branch of solutions, B_L, appears near the lower end of the frequency range. This branch represents the large-amplitude periodic inter-well responses desirable for energy harvesting. It can be clearly seen that, for the range of frequencies considered, this large amplitude branch quickly disappears in a cyclic-fold bifurcation, *cfL*, and gives way to more complex 3-*period* periodic responses that represent a mixture of inter- and intra-well motions. On the other hand, as the frequency is decreased from higher to lower values, it is noted that the cyclic fold bifurcation, *cfB*, occurring on the resonant intra-well branch, B_r, disappears and is replaced by the period doubling bifurcation, *pd*. As the frequency is decreased further, a cascade of period doubling bifurcations occurs leading to a window of inter-well chaotic motions, *CH*, which disappears in a boundary crisis near *cfA*.

When the excitation is increased further to the higher level, $\mathcal{F}=0.165$, as observed in Fig. 2(c), three distinct behaviors are

[1] Approximate analytical methods to predict the complex oscillatory response behavior of bi-stable systems outside the scope of energy harvesting were originally established in [20–23].