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A note on the stability of Beltrami fields for compressible fluid flows

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ABSTRACT

In this paper we study the stability of the magnetostatic equilibrium through a relaxation of a magnetic field **B** in perfectly conducting compressible and viscous fluid.

We establish stability criterion of a large class of Beltrami flows to any admissible displacement about the equilibrium configuration. We show that the field is stable to any displacement with the same 2π-periodicity as the basic flow, except the case where perturbations with wavelength much greater than the scale of the basic flow are included.

1. Introduction

Beltrami fields are 3D divergenceless fields solutions to the equation curl $\mathbf{u}=\alpha\mathbf{u}$, where α is a scalar unknown function. Thus, **u** is eigenfield of the curl operator and by an obvious compatibility condition α is constant along any field line of **u**.

Two situations are distinguished in the literature. One for α a constant function everywhere which characterizes the linear fields and the other is for α a variable function characterizing the non-linear ones. The case $\alpha=0$ corresponds to the wellknown potential theory.

Since their introduction by the Italian mathematician F. Beltrami in 1889 [1] in the study of hydrodynamics, these fields have received considerable attention until the fifties with early work of Bjørgum [2,3], Lundquist [4] and Lust and Schluter [5]. Later they have found applications in many fields of research: plasma physics [6–8], astrophysics specifically in solar physics [9–12], superconductivity [13–15]. In addition, Beltrami fields play an important role in magneto-hydrodynamics, more precisely in the study of kinematic dynamo effects [16–20], where they are known as force-free fields.

In some turbulent flows [21,22], it is shown that velocity and vorticity vectors have a tendency to be aligned in the small scales. This effect is known as local flow Beltramization. Finally, as the Beltrami fields are eigenfunctions of the curl operator, they have generated much interest and numerous mathematical studies ([23–26] and the references therein).

One such class of fields, introduced by Arnold [27], is the ABC flows (named after Arnold, Beltrami, and Childress) which is generated by the velocity field:

$$\mathbf{u}=(A\sin kz+C\cos ky, B\sin kx+A\cos kz, C\sin ky+B\cos kx) \tag{1.1}$$

where (x, y, z) are coordinates on the three-dimensional torus $T^3=R^3/2\pi Z$, and A, B and C are arbitrary constants.

The ABC flows are exact solutions of the Euler equation for an incompressible fluid in steady motion for some suitable pressure. Furthermore, they are specific solutions of the Navier Stokes equations [28,29].

In the case $k=1$, Arnold [27] suggested that the ABC flows have complex topological structures. For particular values of constants A, B and C, Hénon [30] proved that ABC flows exhibit the so-called Lagrangian turbulence; which means that the streamlines of **u** have chaotic behavior. Many developments were later undertaken for general parameters by several authors [31–35]. They argued that streamlines of flows (1.1) are ergodic in a subdomain, in the sense that particle paths of **u** are dense.

2. Statement of the problem

It is well known in Moffatt [36] that the magnetostatic equilibrium with 'ABC' magnetic field

$$\mathbf{B}=(A\sin z+C\cos y, B\sin x+A\cos z, C\sin y+B\cos x) \tag{2.1}$$

has a minimum energy under virtual volume-preserving displacements $\boldsymbol{\eta}(\mathbf{x})$ (i.e. $\nabla.\boldsymbol{\eta}=0$) that convect and distort the field according to the perfect conductivity (frozen-field) induction equation.

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The field (2.1) is a Beltrami field satisfying

$$\nabla \times \mathbf{B} = \mathbf{B} \tag{2.2}$$

In this paper we relax the volume-preserving condition and allow for compressibility and internal energy of the medium. In the following section we will assume that this internal energy depends only on the density. In the fourth section we will consider firstly an incompressible fluid, afterwards we will evaluate the second-order variation $\delta^2 M$ of magnetic energy for arbitrary virtual displacements $\boldsymbol{\eta}$ under the assumption that these admit Fourier decomposition, and that the fluid is at constant temperature.

The energy principle used therein is essentially the one originally introduced by Bernstein et al. [37] with some change in the focus and notations. The actual process of relaxation to a magnetostatic equilibrium can be considered in terms of 'steepest descent' to the minimum energy state. The choice of path is severely constrained when the fluid is incompressible, and is clearly less constrained when the fluid is compressible. This aspect of the problem is discussed in Section 4.

3. Energy functional and its variations

Herein, we consider a compressible infinitely conducting viscous medium. The entropy will be taken as constant. The evolution of this medium is governed by the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0 \tag{3.1}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v}\right) = -\nabla p + \mathbf{J} \times \mathbf{B} + \mu \nabla^2 \mathbf{v} \tag{3.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{3.3}$$

where $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, p is the scalar pressure, μ is the viscosity, $\mathbf{B}$ is the magnetic field and $\mathbf{J} = \nabla \times \mathbf{B}$ is the electric current density.

We assume that there exists an internal density energy per unit mass $\varepsilon(\rho)$ such that

$$\frac{\partial \varepsilon}{\partial \rho} = \frac{p}{\rho^2}. \tag{3.4}$$

The special case of the magnetic relaxation in a barotropic fluid with pressure–density relationship $\varepsilon(\rho) = kp^\gamma$, where $\gamma > 1$ and k are constants, has been studied by Moffatt [38].

Let D be a bounded domain, with fixed boundary ∂D, occupied by the fluid, K the kinetic energy, M the magnetic energy and U the internal energy given by

$$K = \int_D \rho \frac{\mathbf{v}^2}{2}\, dV, \quad M = \int_D \rho \frac{\mathbf{B}^2}{2}\, dV, \quad U = \int_D \rho \varepsilon(\rho) dV \tag{3.5}$$

If the field $\mathbf{B}$ is tangent to ∂D, then it follows that

$$\frac{d}{dt}(K + M + U) = -\mu \int_D \rho |\nabla \mathbf{v}|^2 dV + \int_{\partial D} (\nabla \mathbf{v}.\mathbf{n}).\mathbf{v}\, dS. \tag{3.6}$$

where $\mathbf{n}$ is the outward unit normal vector to ∂D.

In fact we will assume that the surface integrals vanish. This can be achieved either by considering that D is large and that the fields tend to zero in the vicinity of the boundary of D or by assuming that D is a cube and that the fields are periodic functions in D. Therefore under appropriate initial condition, the fluid may relax to an equilibrium state with the associated energy given by the following equation:

$$E = M + U \tag{3.7}$$

Let us show now that the first variation of this energy with respect to any admissible displacement yields the equilibrium equation. This displacement $\boldsymbol{\eta}$ will produce variations of the fields ρ and $\mathbf{B}$ which must be compatible with the constraints (3.1)–(3.3), and with the boundary conditions describing either the periodicity of $\boldsymbol{\eta}$ or from being tangent to the boundary, i.e.

$$\boldsymbol{\eta}.\mathbf{n} = 0. \tag{3.8}$$

Following Moffatt [36], the variations will be

$$\delta \mathbf{B} = \delta^1 \mathbf{B} + \delta^2 \mathbf{B} \tag{3.9}$$

with

$$\delta^1 \mathbf{B} = \nabla \times (\boldsymbol{\eta} \times \mathbf{B}), \quad \delta^2 \mathbf{B} = (1/2) \nabla \times \left(\boldsymbol{\eta} \times \delta^1 \mathbf{B}\right), \tag{3.10}$$

and

$$\delta \rho = \delta^1 \rho + \delta^2 \rho \tag{3.11}$$

with

$$\delta^1 \rho = -\nabla.(\rho \boldsymbol{\eta}), \quad \delta^2 \rho = -(1/2) \nabla.\left(\delta^1 \rho \boldsymbol{\eta}\right) \tag{3.12}$$

Then

$$E(\boldsymbol{\eta}) = E(0) + \delta^1 E + \delta^2 E + O(|\boldsymbol{\eta}|^3) \tag{3.13}$$

with

$$\delta^1 E = \int_D \mathbf{B}.(\delta^1 \mathbf{B} + \delta^1 \rho g'(\rho)) dV \tag{3.14}$$

where $g'(\rho) = d(\rho \varepsilon(\rho))/d\rho$ and

$$\delta^2 E = \frac{1}{2} \int_D \left\{ \left(\delta^1 \mathbf{B}\right)^2 + \mathbf{B}.\delta^2 \mathbf{B} + \left(\delta^1 \rho\right)^2 g''(\rho) + \delta^2 \rho g'(\rho) \right\} dV \tag{3.15}$$

Integrating by part the first variation given by (3.14) yields

$$\delta^1 E = \int_D [\rho \nabla (g'(\rho)) - (\nabla \times \mathbf{B}) \times \mathbf{B})] \boldsymbol{\eta}\, dV + \int_{\partial D} \mathbf{B} \times (\boldsymbol{\eta} \times \mathbf{B}).\mathbf{n}\, dS - \int_{\partial D} \rho g'(\rho) \boldsymbol{\eta}.\mathbf{n}\, dS \tag{3.16}$$

If we take into account the boundary conditions for $\boldsymbol{\eta}$ and $\mathbf{B}$, the surface integrals vanish. But

$g'(\rho) = \varepsilon + (p/\rho)$ due to (3.4), then

$$\nabla g'(\rho) = \frac{d\varepsilon}{d\rho} \nabla \rho + \frac{\nabla \rho}{\rho} - \frac{p}{\rho^2} \nabla \rho = \frac{\nabla \rho}{\rho} \tag{3.17}$$

Therefore

$$\delta^1 E = \int_D [\nabla p - (\nabla \times \mathbf{B}) \times \mathbf{B}].\boldsymbol{\eta}\, dV \tag{3.18}$$

This variation is equal to zero for all admissible displacements if and only if the following equilibrium equation is satisfied:

$$\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{3.19}$$

Consider now the expression for $\delta^2 M$:

$$\delta^2 M = \frac{1}{2} \int_D \left[\left(\delta^1 \mathbf{B}\right)^2 + \mathbf{B}.\nabla \times (\boldsymbol{\eta} \times (\nabla \times (\boldsymbol{\eta} \times \mathbf{B}))\right] dV \tag{3.20}$$

Integration by parts shows that

$$\delta^2 M = \frac{1}{2} \int_D \left[\left(\delta^1 \mathbf{B}\right)^2 - (\boldsymbol{\eta} \times (\nabla \times \mathbf{B})).\delta^1 \mathbf{B} \right] dV \tag{3.21}$$

The second variation of the internal energy is

$$\delta^2 U = \frac{1}{2} \int_D \left[(\nabla.(\rho \boldsymbol{\eta}))^2 g'' + \nabla.(\boldsymbol{\eta} \nabla (\rho \boldsymbol{\eta})) g' \right] dV. \tag{3.22}$$

After integrating by part and taking (3.17) and the equilibrium equation into account, it follows:

$$\delta^2 U = \frac{1}{2} \int_D \left[\left(\delta^1 \rho\right)^2 g'' + \boldsymbol{\eta} \times (\nabla \times \mathbf{B}).\left(\frac{\mathbf{B}}{\rho} \delta^1 \rho\right) \right] dV \tag{3.23}$$

Then the second variation of the total energy is

$$\delta^2 E = \frac{1}{2} \int_D \left[\left(\delta^1 \rho\right)^2 g'' + \boldsymbol{\eta} \times (\nabla \times \mathbf{B}).\left(\frac{\mathbf{B}}{\rho} \delta^1 \rho - 2\delta^1 \mathbf{B}\right) \right] dV \tag{3.24}$$