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The functional constraints method: Application to non-linear delay reaction–diffusion equations with varying transfer coefficients

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ABSTRACT

We present a number of new simple separable, generalized separable, and functional separable solutions to one-dimensional non-linear delay reaction–diffusion equations with varying transfer coefficients of the form

$u_t=[G(u)u_x]_x+F(u,w)$,

where $u=u(x,t)$ and $w=u(x,t-\tau)$, with τ denoting the delay time. All of the equations considered contain one, two, or three arbitrary functions of a single argument. The generalized separable solutions are sought in the form $u=\sum_{n=1}^{N}\varphi_n(x)\psi_n(t)$, with $\varphi_n(x)$ and $\psi_n(t)$ to be determined in the analysis using a new modification of the functional constraints method. Some of the results are extended to non-linear delay reaction–diffusion equations with time-varying delay $\tau=\tau(t)$. We also present exact solutions to more complex, three-dimensional delay reaction–diffusion equations of the form

$u_t=\text{div}[G(u)\nabla u]+F(u,w)$.

Most of the solutions obtained involve free parameters and so may be suitable for solving certain problems as well as testing approximate analytical and numerical methods for non-linear delay PDEs.

1. Introduction. Classes of equations considered. Some remarks

1.1. Non-linear delay reaction–diffusion equations

Non-linear delay reaction–diffusion equations and systems of coupled equations arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, ecology, economics, and many other areas (e.g., see the studies [1–12] and references in them). Similar equations occur in the mathematical theory of artificial neural networks and the results are used for signal and image processing as well as in image recognition problems [13–22].

To begin with, we discuss non-linear delay reaction–diffusion equations of the form

$$u_t=ku_{xx}+F(u,w),\quad w=u(x,t-\tau). \tag{1}$$

The rate of change of the unknown in biochemical, biological, physico-chemical, ecological, and other systems generally depends not only on the current state of the system at a fixed time but also on its entire previous evolution or the values of the unknown at certain times in the past. The latter case is modeled by equations of the form (1), where the kinetic function F (rate of chemical reaction) depends on both $u=u(x,t)$ and its delayed counterpart $w=u(x,t-\tau)$. If $F(u,w)=f(w)$, the delay suggests, from the physical viewpoint, that mass/heat transfer processes in local non-equilibrium media possess inertia: the system does not respond to an action immediately at the time t when the action is applied, which is the case in the classical local-equilibrium case, but a delay time τ later.

The delay τ in reaction–diffusion equations and other non-linear PDEs may be due to a number of different factors depending on the area of application. For example, in biology and biomechanics, the delay may be associated with the finiteness of the speed of neural response in living tissues. In medicine, in decease development problems, the delay time is determined by the incubation period during which the disease develops (in certain cases, one should also take into account the time after which an infected animal becomes contagious). In population dynamics, the delay is due to a gestation or maturation period. In control theory,

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the delay usually results from the finiteness of the signal processing speed and rate of technological processes.

A number of exact solutions to the heat equation with a non-linear source, which is a special case of Eq. (1) where there is no delay and $F(u,w)=f(u)$, are listed, for example, in [23–29]. A comprehensive survey of exact solutions to this class of non-linear equations can be found in the handbook [30]; it also describes a considerable number of generalized and functional separable solutions to non-linear reaction–diffusion systems of two coupled equations without delay.

The presence of delay in Eq. (1) makes it much more difficult to investigate than non-linear PDEs without delay.

In general, Eq. (1) admits traveling-wave solutions, $u=u(\alpha x+\beta t)$. Such solutions are dealt with in many studies (e.g., see the papers [2–7] and references in them). A complete group analysis of the non-linear delay equation (1) was carried out in [11]; four equations of the form (1) were found to admit invariant solutions (two of these equations have degenerate solutions that are linear in x). The studies [12,31–34] describe a number of simple separable, generalized separable, and some other exact solutions to equation (1) as well as several exact solutions to more complex, non-linear reaction–diffusion equations with time-varying delay, $\tau=\tau(t)$, and systems of two coupled delay equations.

The present paper deals with more complex non-linear delay reaction–diffusion equations than (1),

$$u_t=[G(u)u_x]_x+F(u,w), \quad w=u(x,t-\tau), \tag{2}$$

where the transfer (diffusion) coefficient G is dependent on the unknown u.

There are a number of exact solutions obtained for the special case of Eq. (2) where there is no delay and $F(u,w)=f(u)$, which corresponds to a non-linear heat equation; for example, see [23,24,26–30].

The study [35] suggested an exact method based on using invariant subspaces for non-linear ordinary differential operators involved in the equations under consideration [29]. The method allowed the authors to find a few exact solutions to non-linear delay reaction–diffusion equations of the form (2). The scope of applicability of the method is mostly limited to non-linear delay PDEs involving arbitrary parameters (but not arbitrary functions).

The present paper proposes a modified functional constraints method, which generalizes the method of [32] and is more effective than the method of [35]. It serves to construct exact solutions to non-linear delay PDEs involving arbitrary functions. The subsequent sections present are a number of new simple separable, generalized separable, and functional separable solutions to one-dimensional equations of the form (2) (see Sections 3–5), obtained with the proposed method, as well as more complex three-dimensional delay reaction–diffusion equations (see Sections 6–8). Some of the results are extended to non-linear delay reaction–diffusion equations with time-varying delay $\tau=\tau(t)$; see Section 9. All of the equations considered contain one, two, or three arbitrary functions of a single argument.

1.2. The concept of 'exact solution' for non-linear delay PDEs

In what follows, we use the term 'exact solution' with regard to non-linear delay partial differential equations in the following cases [31–33]:

(i) the solution is expressible in terms of elementary functions or in closed form with definite or indefinite integrals;
(ii) the solution is expressible in terms of solutions to ordinary differential or delay ordinary differential equations (or systems of such equations);
(iii) the solution is expressible in terms of solutions to linear partial differential equations.

Combinations of cases (i)–(iii) are also allowed.

This definition generalizes the notion of an exact solution used in [30] with regard to non-linear partial differential equations.

Remark 1. For solution methods and various applications of linear and non-linear delay ordinary differential equations, which are much simpler than non-linear delay partial differential equations, see, for example, [36–41].

Remark 2. Exact solutions to some non-linear delay PDEs (and systems of non-linear delay PDEs) other than (1) and (2) can be found, for example, in [12,42–44].

Remark 3. For numerical solution methods for non-linear coupled delay reaction–diffusion systems and other non-linear systems of delay PDEs as well as related difficulties, see [45–50]. The exact solutions presented below in Sections 3–9 may be used as test problems for independent verification of numerical methods for non-linear systems of delay PDEs.

1.3. Generalized separable and functional separable solutions

In subsequent sections, we will be considering generalized separable solutions of the form

$$u=\sum_{n=1}^{N}\varphi_n(x)\psi_n(t). \tag{3}$$

The functions $\varphi_n(x)$ and $\psi_n(t)$ are to be determined in the course of the analysis using Eq. (2).

The functions $\varphi_n(x)$ appearing in (3) are often chosen from the following forms:

$$\varphi_n(x)=x^m \quad (m=0,1,2), \quad \varphi_n(x)=\exp(\lambda_n x),$$
$$\varphi_n(x)=\cos(\beta_n x), \quad \varphi_n(x)=\sin(\beta_n x) \tag{4}$$

with the parameters λ_n and β_n to be determined from (3) and the original equations. It often suffices to define the functions $\psi_n(t)$ in the same manner.

Remark 4. For non-linear partial differential equations without delay, various modifications of the method of generalized separation of variables based on searching for solutions of the form (3) are detailed, for example, in [28–30]. These studies also present a large number of non-linear PDEs and systems of PDEs that admit generalized separable solutions.

Apart from generalized separable solutions, the paper will also present some other exact solutions, including functional separable solutions of the form

$$u=U(z), \quad z=\sum_{n=1}^{N}\varphi_n(x)\psi_n(t). \tag{5}$$

In the special case $U(z)=z$, solution (5) coincides with (3).

Remark 5. Solutions of the form (5) for various non-linear partial differential equations without delay are considered in [28–30, 51–56].

2. Equations involve arbitrary functions. The functional constraints method

For non-linear delay partial differential equations that involve arbitrary functions, the direct application of the method of generalized separation of variables turns out to be ineffective. It is more reasonable to treat such equations using a modification