International Journal of Non-Linear Mechanics 67 (2014) 278–284

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Modelling the advancement of the impurities and the melted oxygen concentration within the scope of fractional calculus

Abdon Atangana [a,*], Dumitru Baleanu [b,c,d]

[a] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, 9300 Bloemfontein, South Africa
[b] Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box 80204, Jeddah, Saudi Arabia
[c] Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankara University, 06530 Ankara, Turkey
[d] Institute of Space Sciences, Magurele, Bucharest, Romania

ARTICLE INFO

Article history:
Received 7 July 2014
Received in revised form
16 September 2014
Accepted 17 September 2014
Available online 1 October 2014

Keywords:
Rivers pollution
Steady state
Fourier
Laplace and Mellin transform operators
Mittag–Leffler function and H-function
Fractional green function

ABSTRACT

The model describing the mitigation of contamination through ventilation inside a moving waterway polluted via dispersed bases together with connected reduction of liquefied oxygen was investigated within the scope of fractional derivatives. The steady-state cases were investigated using some Caputo derivatives properties. The steady-state solutions in presence and absence of the dispersion were derived in terms of the Mittag–Leffler function. In the case of non-steady state, we derived the solution of the first equation in terms of the α-stable error function via the Laplace transform method. To solve the second equation, we constructed the fractional Green function via the Laplace, Fourier and Mellin transforms. The fractional Green function was expressed by mean of the H-function. Particularly, we presented the selected numerical results a function of distance and α.

1. Introduction

Water contamination is a main international problematic that necessitates ongoing appraisal and reconsideration of water supply strategy at all level. It has been advocated that it is the foremost international foundation of mortality and sicknesses [1,2] and that is accountable for the fatality of more than 14,000 peoples quotidian [2]. Rivers into the oceans eventually carry most water pollutants. For particular area of the globe the guidance can be outlined hundred kilometers since the mouth by investigations employing hydraulic transportation representations [1,2].

The unambiguous impurities controlling the effluence in water comprise but not limited to a widespread range of compounds, pathogens and bodily or sensory variations for example elevation of temperature and discoloration [2]. Whereas countless of chemicals and substances that are synchronized perhaps obviously occurring for example; calcium, sodium, and iron manganese and so on, the concentration however is habitually the main point in determining whatever a natural component of water is, and whatever a pollution will be. It is perhaps important to recall that, elevated concentrations of instinctively appearing substances can have negative influences upon marine vegetation and fauna. Many others studies in connection of the movement of flow within porous media have been done recently in [20–24].

Scientific water excellence models can be traced since the year 1920s during the year 1925, the distinguished model proposed in [3] portrayed the stability of liquefied oxygen in the waterway. Consequently this model has been altered in numerous techniques see for instance in [4]. One of the main goals of this investigation was to study the mitigation of contamination through ventilation inside a moving waterway dirtied with dispersed bases and the connected reduction of liquefied oxygen. It is important to note that, natural environmental dumps by means of exceedingly divergent permeability may form mobile and relatively motionless regions, where the potential mass exchange connecting mobile and stationary regions marks in a wide time distribution for solute usually referred as "trapping" [28]. The transport progression, shared with the dissimilar particle status, can be considered powerfully by the time-non-local model, including the time meaning the use of non-integer order derivative [28]. If the high-permeable material tends to form preferential flow paths, such as the interconnected paleochannels observed in alluvial depositional systems, then the solute transport may show a heavy leading edge, which can be described by the space fractional with maximally positive skewness [28]. We shall in this work carry investigation on the generalization of the flow within the waterway as being a one-dimensional, employing the solitary space parameter x to portray the distance down the waterway from its base to the concept of

* Corresponding author.
E-mail addresses: abdonatangana@yahoo.fr (A. Atangana), dumitru@cankaya.edu.tr (D. Baleanu).

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.09.010
0020-7462/

fractional calculus. Magnitudes, for example impurity or oxygen absorption, are only endorsed to fluctuate lengthways the piece of the waterway also they are treated as homogeneous transversely the waterway cross-section [5]. Also, the reason of this generalization is due to the fact that, many studies have indicated that it is preferable modeling physical problems with fractional order derivative rather than ordinary derivative [6–10]. On the other hand, many studies have been done regarding the anomalous diffusion with linear reaction dynamics from continuous time random walk to fractional reaction–diffusion equations [24–27]. The model under study here will be self-possessed of two-combined advection-dispersion equation these mathematical formulas will comprise the development of impurities together with liquefied oxygen concentrations respectively. Rate of variation for these measurements with the position of x and time t are conveyed in mathematical equation as

$$\begin{aligned} {}_0^C D_t^\alpha [A\,C_1(x,t)] &= D_1 \frac{\partial^2}{\partial x^2}[A\,C_1(x,t)] - {}_0^C D_x^\alpha [Av\,C_1(x,t)] - k_1 \frac{C_2(x,t)}{C_2(x,t)+k} AC_1(x,t) \\ &\quad + qH(x), 0<\alpha\le 1 \\ {}_0^C D_t^\alpha [A\,C_2(x,t)] &= D_2 \frac{\partial^2}{\partial x^2}[A\,C_2(x,t)] - {}_0^C D_x^\alpha [Av\,C_2(x,t)] \\ &\quad - k_2 \frac{C_2(x,t)}{C_2(x,t)+k} AC_1(x,t) + \beta(S - C_2(x,t))\ (x<L,\quad t>0) \end{aligned} \tag{1}$$

where L is the waterway length measured in meter, D_1 and D_2 are the dispersion coefficients of pollutant and dissolved oxygen respectively, v is the fluid velocity in the x-direction, A is the area of waterway cross-section, q is the rate of pollutant additional along the waterway, k_1 and k_2 are the degradation and de-aeration rate coefficients of pollution and oxygen at 20 °C respectively; k is the half-saturated oxygen demand for pollution concentration for pollution decay; β is the mass transfer of oxygen from air to water [1]; S is the saturated oxygen concentration, $H(x)$ is the Heaviside function and finally ${}_0^C D_t^\alpha[]$ is the fractional derivative according to Caputo. To familiarize readers that are not in the field of fractional calculus, we present in the next section, brief information about the fractional calculus, and more importantly some useful properties of the Caputo fractional derivative. This model divides the river in two, for $x<0$ we have the upstream near the source of the pollution wherever we presumed that there no additional contamination and for $x>0$ we have the downstream where pollution is added at a rate q.

2. Some useful information's about fractional derivative

We devote this section to the discussion underpinning the basic ideal of the fractional calculus. But we will much stress in the properties of Caputo derivative since it will be used throughout the remaining of the paper.

Definition 1. Riemann–Liouville integral [11–13].

The Riemann–Liouville integral, contributes the conventional form of fractional calculus. The concept of intermittent functions consequently comprising the boundary a condition of iterating is referred the Weyl integral. The Riemann–Liouville integral of order α is given as

$${}_a D_t^{-\alpha} f(x) = {}_a I_t^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt \tag{2}$$

Definition 2. Caputo fractional derivative.

Here is additional on behalf of calculating fractional derivatives; the Caputo fractional derivative. Caputo initiated this version in his work in the year 1967 [14]. In contradiction of the Riemann–Liouville fractional definition, while resolving differential equation employing Caputo's definition, one does not need to describe the initial conditions in fractional derivative. However Caputo's definition is exemplified as follows:

$${}_0^C D_t^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt \tag{3}$$

The above definitions are commonly used in pure and applied mathematics.

Definition 3. The Laplace transform is an extensively employed integral transform with countless presentations in physics and engineering. Thus the Laplace transform of a given function says g is provided by

$$\mathcal{L}(f(x))(s) = \int_0^\infty e^{-sx} f(x)dx \tag{4}$$

Let us observe that the Laplace transform of fractional derivative with Caputo.

$$\mathcal{L}({}_0^C D_t^\alpha f(x))(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),\ (n-1<\alpha\le n) \tag{5}$$

The above use the usual initial conditions or values of the functions.

Another useful property of the Caputo derivative is the following:

$${}_a D_t^\alpha [{}_0^C D_t^\alpha f(x)] = f(x) - \sum_{j=1}^{n-1} \frac{f_j(0)}{\Gamma(\alpha-j+1)} x^j,\ (n-1<\alpha\le n) \tag{6}$$

here Γ is the gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \tag{7}$$

3. Analysis and solutions

Non-linear fractional partial differential equation is sometime very difficult to handle analytically. In some cases, further simplifications are needed in order to reduce the complexity of the problem. Therefore, we first study several exceptional situations aimed at which the equations are relatively reduced and can perhaps be handled with analytical methods and then portray an introductory mathematical methodology to resolving the to solving the more universal problem. Presently we shall for some reasons get rid of the restriction $x<L$, because of the properties of the physical problem under investigation. We shall start with the case where the model is steady state, meaning the time is ignored.

3.1. Steady state case in the absence of dispersion

An interesting fact with the Caputo fractional derivative is that, his derivative of all real numbers is zero. Therefore if we assume that the concentrations of the species under investigation are time independent, we have that

$${}_0^C D_t^\alpha [A\,C_1(x,t)] = {}_0^C D_t^\alpha [A\,C_2(x,t)] = 0, \tag{8}$$

we assume that, there is no dispersion along the downstream for the first and the second pollution then system of Eq. (1) is reduced to

$$\begin{aligned} {}_0^C D_t^\alpha [Av\,C_1(x)] &= -k_1 \tfrac{C_2(x)}{C_2(x)+k} AC_1(x) + qH(x) \\ {}_0^C D_x^\alpha [Av\,C_2(x)] &= -k_2 \frac{C_2(x)}{C_2(x)+k} AC_1(x) + \beta(S - C_2(x)) \end{aligned} \tag{9}$$

here we consider the initial condition to be $C_1(0)=0,\ C_2(0)=S$. Obviously in this situation, we assume not source of effluence upstream due to the missing of dispersion, therefore we have the exact solution of Eq. (9) to be $C_1(0)=0,\ C_2(0)=S$.