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Progressive waves in non-ideal gases

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ABSTRACT

Using the theory of progressive waves and some related procedures, waves of finite and moderately small amplitudes, influenced by the effects of non-linear convection, attenuation and geometrical spreading are studied in an imperfect gas modeled by the van der Waals equation of state; conditions within the wave region, which lead to a shock or no shock depend strongly on the van der Waals gas-parameters. A few specific cases are considered to trace out a complete history of shock decay after its formation. Asymptotic decay laws for perfect gases are exactly recovered.

1. Introduction

A large number of physical processes are described by means of mathematical models represented by quasilinear hyperbolic systems of partial differential equations (see [1–5]). Several methods of approach to investigate the asymptotic properties of weakly non-linear waves governed by hyperbolic systems have been developed which give rise to a transport equation describing the wave process asymptotically (see [6–9]). The study of progressive waves has received great attention in the past as it was greatly motivated by the sonic boom problem in aerodynamics. The major contributions in the theory of progressive waves for linear systems, which are useful as an introduction for the non-linear case, may be found in [10]. Besides its applications in fluid dynamics, it has been applied with great success to various problems in non-linear acoustics, non-linear elasticity, magnetohydrodynamics and other branches of mechanics. The essential ideas underlying the theory of progressive waves may be found in [11–15] under the title ‘Relatively undistorted waves’; a parallel attempt has been made by Taniuti, Asano and their co-workers under the general title ‘Reductive Perturbation Method’ (see [16–18]). It is well known that non-linearity decisively alters the character of both expansion and compression waves in the near and far fields in the sense that disturbances forget the details of their earlier history and remember only the global initial condition such as the total energy input. The theory of progressive waves, in contrast to the theory of non-linear geometrical acoustics which deals with small amplitude waves, deals with finite amplitude pulses; it is exact for Riemann waves, acceleration waves and for the formation of shocks. In the far field, it produces asymptotic expansions for flow variables. In the present paper, we study the planar and non-planar waves of finite and moderately small amplitudes in an imperfect gas modeled by van der Walls equation of state to understand how the real gas effects influence certain features of shock wave propagation. Specific cases in which the initial disturbance is either a pulse or a periodic wave are considered for tracing out the early history of shock decay after its formation; the asymptotic decay laws for weak shocks are obtained along with the real gas effects that influence the evolutionary behavior of waves as they propagate. In this connection, we refer to a recent paper by Zhao et al. [19], which reinforces the fact that shock waves in a van der Waal's fluid exhibit a richer behavior than that predicted by the ideal gas model, characterizing compressive shocks, rarefaction shocks, and shock splitting phenomena together with their admissibility; for the physical meaning of van der Waals gas and its influence on wave motion, we refer to [20–22].

2. Preliminaries

We consider disturbances in a one dimensional flow of a more general class of real gases whose equation of the state [20–22] is governed by

$$(p+a\rho^2)(1-b\rho)=RT\rho, \tag{1}$$

where p is the pressure of the gas particle, ρ is the density, T is the temperature and R is the universal gas constant representing the van der Waal gases. Here, the constant a denotes the amount of attraction between each particle that leads to added pressure due to intermolecular forces of attraction and the constant b denotes the omitted volume and is related to the volume of the gas. It is

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observed that a gas behaves like a real gas at low temperatures and high pressures.

For the given equation of state (1) of a gas, the internal energy e, in view of $R=(\gamma-1)C_V$ where C_V specific heat at constant volume, is given by

$$e=\frac{(p+a\rho^2)(1-b\rho)-a(\gamma-1)\rho^2}{(\gamma-1)\rho}.$$

Here γ is a constant. In general, for a real gas, internal energy depends on the pressure p and density ρ. However, in an ideal gas, i.e., when $a=0$ and $b=0$, the internal energy e for the given flow variables becomes a function of p/ρ; equivalently internal energy for an ideal gas depends only on the temperature and then γ turns out to be the specific heat ratio of an ideal gas.

The basic equations governing a planar or a radially symmetric flow of a compressible fluid, whose equation of state is given by Eq. (1), can be written in the following form:

$$\rho_t+u\rho_x+\rho u_x+\frac{m\rho u}{x}=0,$$
$$u_t+uu_x+\frac{p_x}{\rho}=0,$$
$$p_t+up_x+\rho A^2\left(u_x+\frac{mu}{x}\right)=0. \tag{2}$$

Here the sound speed is given by $A=\sqrt{(\gamma p+a\rho^2(\gamma-2+2b\rho))/\rho(1-b\rho)}$ where x is the spatial coordinate (being either axial in flows with planar $(m=0)$ geometry or radial in cylindrically symmetric $(m=1)$ or spherically symmetric $(m=2)$ flows), t is the time and u is the gas velocity along x- axis. The flow variables with a letter subscript x and/or t denote partial differentiation with respect to the indicated variable.

It may be noticed that the thermodynamic stability of the flow under consideration requires that

$$\gamma p+a\rho^2(\gamma-2+2b\rho)>0, \tag{3}$$

which indeed renders the system (2) to be hyperbolic.

3. Progressive waves

The solution of the system (2) is said to describe a progressive wave solution if there exists a family of wavelets $\phi(x,t)=c=$constant such that the variation of flow variables ρ, u and p with respect to x, for any fixed wavelet $\phi=c$, is much less than the variation of the flow variables with respect to x for a fixed time t. Such a motion is obviously an extension of the concept of a simple wave, where one can find a variable $\phi(x,t)$ such that the flow variables can be expressed in terms of ϕ only; in other words, the flow variables remain constant if, and only if, one stays on the wavelet. This suggests that the progressive waves, considered here, can be regarded as slowly modulated simple waves. Thus, to determine a progressive wave solution, we consider a transformation from (x,t) to (x,ϕ) through $t=\tau(x,\phi)$. Then the basic equations (2) in terms of the $\rho(x,t)=\overline{\rho}(x,\phi)$, $u(x,t)=\overline{u}(x,\phi)$ and $p(x,t)=\overline{p}(x,\phi)$ reduce to the following form:

$$(1-u\tau_x)\rho_t-\rho u_t\tau_x+\overline{u}\,\overline{\rho}_x+\overline{\rho}\,\overline{u}_x+\frac{m\overline{\rho}\,\overline{u}}{x}=0,$$
$$(1-u\tau_x)u_t-\frac{1}{\rho}\tau_x p_t+\overline{u}\,\overline{u}_x+\frac{1}{\overline{\rho}}\overline{p}_x=0,$$
$$(1-u\tau_x)p_t-\rho A^2\tau_x u_t+\overline{u}\,\overline{p}_x+\overline{\rho}\overline{A}^2\left(\overline{u}_x+\frac{m\overline{u}}{x}\right)=0, \tag{4}$$

where $\overline{A}=\sqrt{(\gamma\overline{p}+a\overline{\rho}^2(\gamma-2+2b\overline{\rho}))/\overline{\rho}(1-b\overline{\rho})}$, and since the solution is considered to be a progressive wave, we have

$$|\overline{\rho}_x|\ll|\rho_x|\Rightarrow \rho_x\simeq\rho_t\tau_x\Rightarrow|\overline{\rho}_x|\ll|\rho_t|,$$
$$|\overline{u}_x|\ll|u|\Rightarrow u_x\simeq u_t\tau_x\Rightarrow|\overline{u}_x|\ll|u_t|,$$
$$|\overline{p}_x|\ll|p|\Rightarrow p_x\simeq p_t\tau_x\Rightarrow|\overline{p}_x|\ll|p_t|,$$

and if $\overline{u}_x=O(\overline{u}/x)$, then Eqs. (4) can be written in the following form:

$$(1-u\tau_x)\rho_t-\rho u_t\tau_x=0,$$
$$(1-u\tau_x)u_t-\frac{1}{\rho}\tau_x p_t=0,$$
$$(1-u\tau_x)p_t-\rho A^2\tau_x u_t=0, \tag{5}$$

which lead to

$$\tau_x=(u+A)^{-1}, \tag{6}$$

implying thereby that the wavelets are the characteristic curves of (2). The system (5), in view of (6), can be written as

$$\overline{p}_\phi=\overline{A}^2\overline{\rho}_\phi=\overline{\rho}\overline{A}\overline{u}_\phi. \tag{7}$$

Multiplying Eq. $(4)_2$ by $\overline{\rho}\,\overline{A}$ and adding to Eq. $(4)_3$ gives a compatibility condition involving $\overline{\rho}$, $\overline{u}$, $\overline{p}$ and in their derivatives as

$$(\overline{u}+\overline{A})\left(\overline{\rho}\overline{A}\overline{u}_x+\overline{p}_x\right)+\frac{m\overline{\rho}\,\overline{u}\overline{A}^2}{x}=0. \tag{8}$$

4. Finite amplitude disturbance

Here, we consider the disturbance propagating into a uniform region $\rho=\rho_0$, $u=0$ and $p=p_0$. As it is possible to label the wavelets, we let the boundary conditions for $\overline{\rho}$ and τ to be

$$\overline{\rho}=g(\phi),\quad \tau=\phi,\quad \text{at } x=x_0, \tag{9}$$

where g is a smooth bounded function, i.e., $|g|=O(1)$. In the progressive wave approximation, it follows from (7), that

$$\overline{p}(\phi,x)=P(\overline{\rho}(\phi,x)),\quad \overline{u}(\phi,x)=U(\overline{\rho}(\phi,x)). \tag{10}$$

In view of (10), Eqs. (6) and (8) can be solved for $t=\tau(x,\phi)$ and $\overline{\rho}(x,\phi)$ respectively as functions of x and ϕ as

$$\tau=\phi+\int_{x_0}^{x}\frac{1}{U(\overline{\rho})+F(\overline{\rho})}\,dx,\quad \overline{\rho}U(\overline{\rho})=G(\phi)(x/x_0)^{-m/2}, \tag{11}$$

where $G(\phi)=g(\phi)U(g(\phi))$ and

$$P(\overline{\rho})=\left(\frac{\overline{\rho}}{1-b\overline{\rho}}\right)^\gamma\left(p_0\left(\frac{1-b\rho_0}{\rho_0}\right)^\gamma+a\int_{\rho_0}^{\overline{\rho}}\left(\frac{(\gamma-2+2bs)s^{1-\gamma}}{(1-bs)^{1-\gamma}}\right)ds\right),$$
$$U(\overline{\rho})=\int_{\rho_0}^{\overline{\rho}}\frac{F(s)}{s}\,ds,$$
$$F(s)=\sqrt{\frac{\gamma P(s)+as^2(\gamma-2+2bs)}{s(1-bs)}}.$$

Eq. (11) indicates that the shock first forms at $x=x_s$ on the wavelet ϕ_s, where x_s is a solution of

$$1+\int_{x_0}^{x_s}\left(\frac{\gamma(\gamma+1)P(\overline{\rho})+a\overline{\rho}^2(\gamma^2+\gamma-6+12b\overline{\rho}-6b^2\overline{\rho}^2)}{2\overline{\rho}^2(1-b\overline{\rho})^2F(\overline{\rho})(U(\overline{\rho})+F(\overline{\rho}))^2}\right)\frac{\partial\overline{\rho}}{\partial\phi}\Big|_{\phi=\phi_s}dx=0. \tag{12}$$

Eqs. (10)–(12) constitute the desired modulated simple wave solution; indeed, the disturbance that propagates into the uniform region and is described by Eqs. (10)–(12), is determined from Eqs. (11) and then the density $\overline{\rho}$ and subsequently the pressure $\overline{p}$ and velocity $\overline{u}$ are determined from (10). It is also evident from (12) that the solution may break down after a finite running length x_s depending on the values of a and b; from this point onward, we have to envisage a shock wave that