International Journal of Non-Linear Mechanics 67 (2014) 342–351

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

A multiple scale time domain collocation method for solving non-linear dynamical system

Honghua Dai [a,*], Xiaokui Yue [a,*], Cheinshan Liu [b]

[a] College of Astronautics, Northwestern Polytechnical University, Xi'an 710072, China
[b] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

ARTICLE INFO

Article history:
Received 25 April 2014
Received in revised form
17 July 2014
Accepted 1 October 2014
Available online 12 October 2014

Keywords:
Multiple scale time domain collocation method
Duffing equation
Ill-posedness
Multiple scale differential transformation method

ABSTRACT

Recently, a simple time domain collocation (TDC) method was proposed by researchers including the first author of the present paper, and has been successfully applied to obtain the harmonic, subharmonic, and superharmonic responses of the nonlinear Duffing oscillator. The TDC method is based on the point-collocation method performing within an appropriate period of the periodic solution, wherein the approximate solution is assumed as a Fourier series. Upon using the TDC method, the ordinary differential equation is transformed into a system of non-linear algebraic equations (NAEs), which can be readily solved by an NAE solver. In this study, using the Duffing oscillator as the prototype, we develop a multiple scale time domain collocation (MSTDC) method, by introducing a series of optimal multiple scales to the Fourier series of the approximate solution, to alleviate the ill-posedness of the system of collocation-resulting NAEs due to the inclusion of very high order harmonics in the approximate solution. Besides the MSTDC method, a multiple scale differential transformation (MSDT) method is proposed by introducing the multiple scales to the classical differential transformation method. Finally, numerical experiments are carried out to verify the accuracy and efficiency of the MSTDC and the MSDT methods.

1. Introduction

Due to the existence of non-linearity, very rarely can the exact analytical solution of a non-linear problem be obtained [2]. The study of non-linear oscillators is of growing interest to many researchers in the dynamics community [3]. In the literature, there are a variety of approximate methods capable of solving non-linear ordinary equations (ODEs), among which the numerical integration method is the most straightforward one. However, this method is extremely time consuming for large-scale systems, and not convenient for parametric variations. Hence, effective alternatives are necessary.

Many semi-analytical methods have been developed for solving the periodic solutions of the non-linear oscillators. Some representatives are the perturbation methods [4], the differential transformation method [5], the harmonic balance (HB) method [6,7], the high dimensional harmonic balance (HDHB) method [8,9], and the time domain collocation (TDC) method [1]. In Dai et al. [1], the TDC method was initially developed as a simple alternative method to solve non-linear oscillatory problems, wherein it was successfully applied to investigate the complex responses of the Duffing oscillator. More importantly, Dai et al. [1] found that the TDC method was mathematically equivalent to the HDHB method, and pointed out that the HDHB method is not a modified harmonic balance method, but essentially a TDC method in disguise. Dai et al. [10] extended the TDC method via increasing the number of collocation points to successfully eliminate the fake solutions which may arise from the HDHB solution procedure. Furthermore, Dai et al. [7] conducted a comparative study of the classical HB method, the TDC method, and the HDHB method in the framework of solving a two dimensional airfoil in subsonic flow, wherein they clearly explained the aliasing mechanism of the HDHB method.

In this study, we extend the TDC method to the multiple scale time domain collocation (MSTDC) method by modifying the Fourier-series-type trial functions in the TDC method. The multi-scaling concept and the equilibrated matrix were introduced by Liu and Atluri [11] in solving a Laplacian Cauchy problem. In mathematics, a Fourier series decomposes periodic functions into a set of simple oscillating functions, namely sines and cosines $x(t)=a_0+\sum_{k=1}^{m}(a_k \cos k\omega t+b_k \sin k\omega t)$. Theoretically, if m is large enough, the approximation $x(t)$ can be as accurate as desired. However, in practical applications, it is not a good method to simply increase the number of harmonics to obtain a high accurate approximation, because the problem will become ill-posed when the number of harmonics is large. Due to the above reason, the Fourier series with very high order harmonics is rarely used to approximate a solution.

* Corresponding authors.
E-mail addresses: daihonghua@gmail.com (H. Dai), xkyue@nwpu.edu (X. Yue).

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.10.001
0020-7462/© 2014 Elsevier Ltd. All rights reserved.

In order to alleviate the ill-posedness induced by high order harmonics, we scale the terms $\cos k\omega t$ and $\sin k\omega t$ to $\cos(k\omega t)/R_{2k-1}$ and $\sin(k\omega t)/R_{2k}$, by introducing proper scales R_{2k-1}, R_{2k}, which are labeled as the multiple scales.[1] Many efforts have been made towards obtaining appropriate scales so as to reduce the ill-conditionness in the polynomial interpolation as much as possible. Liu and Atluri [12] introduced a characteristic length R_0 into the high-order polynomial expansion, which improved the numerical accuracy for the applications to solve some ill-posed linear problems. Subsequently, Liu and Atluri [13] used multi-scales R_k to precondition the Vandermonde matrix, and achieved a better result than the single scale method. Nevertheless, the multi-scales R_k cannot be obtained in a systematical way.

In this paper, we derive a set of optimal multiple scales for the Fourier series interpolation based on the equilibration of matrices. The modified Fourier series with optimal multiple scales is named the multiple scale Fourier series (MSFS). The TDC method in conjunction with the MSFS is referred to as the multiple scale time domain collocation (MSTDC) method. The collocation-resulting NAEs by using the MSTDC is named MSTDC algebraic system. The MSTDC method can be applied to solve different types of non-linear oscillatory problems. Compared with the TDC method, the MSTDC method can make the collocation-resulting NAEs converge faster. We also extend the differential transformation method by introducing the multi-scaling, and propose the multiple scale differential transformation (MSDT) method.

The structure of this paper is as follows. The MSTDC is introduced in Section 2, and the MSDT is proposed in Section 3. In Section 4, the convergence property of the MSTDC algebraic system is examined. Section 5 verifies the accuracy of the MSTDC and the MSDT by comparing with the conventional TDC method, the differential transformation method, and the harmonic balance method. Conclusions are drawn in Section 6.

2. Multiple scale time domain collocation (MSTDC) method

2.1. Fourier series interpolation

In mathematics, trigonometric interpolation is interpolation with trigonometric polynomials. Interpolation is the process of finding a function which goes through some given data points. For trigonometric interpolation, this function has to be a trigonometric polynomial, that is, a sum of sines and cosines of given periods. Since the trigonometric polynomial has a periodic form, it is naturally well suited for approximating periodic functions.

Given a set of $n=2m+1$ data points, $(\hat{t}_i, \hat{y}_i)$ where no two $\hat{t}_i$ are the same, one is looking for a trigonometric polynomial $x(t)$ of order m with the following property:

$$x(\hat{t}_i)=\hat{y}_i, \quad i=1,\dots,n, \tag{1}$$

where $\hat{t}_i \in [a,b]$, and $[a,b]$ is the problem domain. Traditionally, for implementing the Fourier interpolation, one will introduce the following truncated Fourier series $x(t)$ to interpolate the given data points:

$$x(t)=a_0+\sum_{k=1}^{m}(a_k \cos k\omega t+b_k \sin k\omega t), \tag{2}$$

where 1, $\cos k\omega t$, $\sin k\omega t$, $\{k=1,2,\dots,m\}$, constitute a set of trigonometric basis. Upon substituting Eq. (2) into Eq. (1), we can obtain a system of linear equations to determine the coefficients $a_0, a_k, b_k, \{k=1,2,\dots,m\}$, which in a matrix form reads

$$\begin{bmatrix} 1 & \cos\omega\hat{t}_1 & \sin\omega\hat{t}_1 & \dots & \cos m\omega\hat{t}_1 & \sin m\omega\hat{t}_1 \\ 1 & \cos\omega\hat{t}_2 & \sin\omega\hat{t}_2 & \dots & \cos m\omega\hat{t}_2 & \sin m\omega\hat{t}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos\omega\hat{t}_{2m} & \sin\omega\hat{t}_{2m} & \dots & \cos m\omega\hat{t}_{2m} & \sin m\omega\hat{t}_{2m} \\ 1 & \cos\omega\hat{t}_{2m+1} & \sin\omega\hat{t}_{2m+1} & \dots & \cos m\omega\hat{t}_{2m+1} & \sin m\omega\hat{t}_{2m+1} \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m \end{bmatrix} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_{2m} \\ \hat{y}_{2m+1} \end{bmatrix}, \tag{3}$$

where the real numbers $\{\hat{t}_1,\dots,\hat{t}_n\}$ are called interpolating nodes. We are required to solve the linear system (3) for the coefficients to recover $x(t)$ expressed in Eq. (2).

2.2. Multiple scale Fourier series

Theoretically, the Fourier series of Eq. (2) can approach a continuous periodic function as accurate as possible provided that the number of harmonics is large enough. However numerically the problem of solving Eq. (3) for the unknown coefficients $a_0, a_k, b_k, \{k=1,2,\dots,m\}$, is ill-posed due to the inclusion of very high order harmonics. This is a serious drawback of the Fourier interpolation.

In order to mitigate the ill-posedness, we herein re-construct the above linear system by introducing a series of multiple scales R_{2k-1}, R_{2k} in the Fourier series. We propose the following multiple scale Fourier series (MSFS):

$$x(t)=a_0+\sum_{k=1}^{m}\left(\frac{a_k}{R_{2k-1}}\cos k\omega t+\frac{b_k}{R_{2k}}\sin k\omega t\right), \tag{4}$$

Then, we use the MSFS to interpolate the given data points, and obtain a new system of linear equations

$$\begin{bmatrix} 1 & \frac{\cos\omega\hat{t}_1}{R_1} & \frac{\sin\omega\hat{t}_1}{R_2} & \dots & \frac{\cos m\omega\hat{t}_1}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_1}{R_{2m}} \\ 1 & \frac{\cos\omega\hat{t}_2}{R_1} & \frac{\sin\omega\hat{t}_2}{R_2} & \dots & \frac{\cos m\omega\hat{t}_2}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_2}{R_{2m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{\cos\omega\hat{t}_{2m}}{R_1} & \frac{\sin\omega\hat{t}_{2m}}{R_2} & \dots & \frac{\cos m\omega\hat{t}_{2m}}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_{2m}}{R_{2m}} \\ 1 & \frac{\cos\omega\hat{t}_{2m+1}}{R_1} & \frac{\sin\omega\hat{t}_{2m+1}}{R_2} & \dots & \frac{\cos m\omega\hat{t}_{2m+1}}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_{2m+1}}{R_{2m}} \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m \end{bmatrix} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_{2m} \\ \hat{y}_{2m+1} \end{bmatrix}. \tag{5}$$

Hence, we can solve the above re-constructed linear algebraic system for a_0, a_k and b_k $\{k=1,2,\dots,m\}$ to recover the new Fourier series $x(t)$ in Eq. (4). The coefficient matrix of the above linear system is

$$\mathbf{D}=\begin{bmatrix} 1 & \frac{\cos\omega\hat{t}_1}{R_1} & \frac{\sin\omega\hat{t}_1}{R_2} & \dots & \frac{\cos m\omega\hat{t}_1}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_1}{R_{2m}} \\ 1 & \frac{\cos\omega\hat{t}_2}{R_1} & \frac{\sin\omega\hat{t}_2}{R_2} & \dots & \frac{\cos m\omega\hat{t}_2}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_2}{R_{2m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{\cos\omega\hat{t}_{2m}}{R_1} & \frac{\sin\omega\hat{t}_{2m}}{R_2} & \dots & \frac{\cos m\omega\hat{t}_{2m}}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_{2m}}{R_{2m}} \\ 1 & \frac{\cos\omega\hat{t}_{2m+1}}{R_1} & \frac{\sin\omega\hat{t}_{2m+1}}{R_2} & \dots & \frac{\cos m\omega\hat{t}_{2m+1}}{R_{2m-1}} & \frac{\sin m\omega\hat{t}_{2m+1}}{R_{2m}} \end{bmatrix}. \tag{6}$$

$\mathbf{D}$ is the coefficient matrix of the multiple scale Fourier series

[1] One should not confuse the multiple scales R_{2k-1}, R_{2k} with the multiple scale method which is a type of perturbation method.