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Bifurcation diagram of the self-sustained oscillation modes for a system with dynamic symmetry☆

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ABSTRACT

An autonomous dynamical system with one degree of freedom is considered which possesses properties such that an asymptotically stable equilibrium becomes unstable after a certain parameter passes through zero and two new symmetrically arranged equilibria are created alongside it. It is known that, for sufficiently small values of the above mentioned parameter, bifurcation can be accompanied by the occurrence of periodic trajectories (cycles). To describe them, a bifurcation diagram of the relation between the amplitude of the cycles and the parameter, which characterizes the dissipation and takes finite values, is constructed. The results obtained are illustrated using the example of an investigation of the self-induced oscillatory modes in a model of an aerodynamic pendulum that takes account of the displacement of the pressure centre when the angle of attack is changed.

During the investigation of a certain mathematical model [1,2] of the oscillations of an aerodynamic pendulum, which is a system of differential equations and transcendental algebraic equations, it was discovered that a special bifurcation occurred when a parameter passed through a specific value for which, instead of a stable equilibrium position, a certain structure arose consisting of several stationary points and an orbitally stable cycle surrounding these points. The mathematical problems associated with a bifurcation of this type have been considered.[3,4] It is of interest to explain in greater detail the mechanism of changeover in the character of the behaviour of an aerodynamic pendulum that corresponds to such an unusual bifurcation and to investigate precisely which force actions are responsible for this phenomenon. This note is concerned with this.

1. Statement of the problem

We consider an oscillatory mechanical system (of the pendulum type) with one degree of freedom when there are conservative forces and positional-viscous friction forces [5] that depend on a certain parameter that we henceforth treat as the bifurcation parameter of the model. Suppose the behaviour of such an object is described by the differential equations

$$\dot{\varphi} = \omega, \quad \dot{\omega} = -h(\varphi, s)\omega - q(\varphi, s) \tag{1.1}$$

where s is a parameter which takes values from a certain compact set $S \subset R^{(1)}$, and $h(\varphi, s)$ and $q(\varphi,s)$ are sufficiently smooth function of their arguments.

The term $h(\varphi, s)\omega$ describes the "positional-viscous" dissipative/antidissipative forces and the function $q(\varphi,s)$ describes the positional forces (for example, the restoring or overturning aerodynamic torque).

We shall assume that, for all $s \in S$, system (1.1) possesses the property of dynamic symmetry, that is, for every solution $\varphi(t)$ of Eqs (1.1), the function $-\varphi(t)$ is also a solution (physically, this means that precisely the same backward motion corresponds to every forward motion of the pendulum). In particular, for any $s \in S$ there is a steady solution of system (1.1)

$$\varphi(t) \equiv \varphi^0 = 0 \tag{1.2}$$

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The corresponding fixed point (the quiescent point) of the phase plane (φ, ω) is denoted by O. The above symmetry means that the functions $h(\varphi,s)$ and $q(\varphi,s)$ possess the following properties:

$$h(\varphi,s)=h(-\varphi,s),\ \ q(\varphi,s)=-q(-\varphi,s),\ \ q(0,s)=0.$$

We expand them in Taylor's series with respect to φ with an accuracy up to third order terms and, with respect to s, with an accuracy up to the first order. Taking account of the properties of these functions, we obtain

$$h(\varphi,s)\cong a_0+as+b\varphi^2,\quad q(\varphi,s)\cong(c_0+cs+d\varphi^2)\varphi \tag{1.3}$$

We subsequently assume that the coefficients a, b, c and d are positive, and a_0 and c_0 are non-negative. Suppose the character of the stability of point O changes when the bifurcation parameter s passes through a zero value, that is, just one of the coefficients a_0 or c_0 is equal to zero. Then, when $s>0$, this point is stable and, when $s<0$, it is unstable. We now consider issues associated with the loss of stability of point O in greater detail.

Different mechanisms of the loss of stability when there is a local change in the properties of the functions $h(\varphi,s)$ and $q(\varphi,s)$ near the point O are possible.

1. Due to the conversion of dissipative forces into antidissipative forces (a so-called "soft" loss of stability [3]). In this case, the quantity $h(0,s)$ changes sign when s passes through a zero value and $q'(0, s)$ does not change sign (here and subsequently, a derivative with respect to φ is denoted by a prime). This version of stability loss is associated with a standard Andronov–Hopf bifurcation when a stable cycle borns from the stable point O, encompassing this point that has become unstable, and the "amplitude" of the cycle is proportional to the quantity $\sqrt{s}$.[3] In this case, $a_0=0$ and $c_0>0$.
2. Due to a change in the character of the positional part of the force action (from a restoring into an overturning force). In this case, the quantity $q'(0, s)$ changes sign when the bifurcation parameter s passes through zero and the sign of the function $h(0,s)$ is preserved, that is, $a_0>0$, $c_0=0$ and, in addition to the point O, a further two new fixed points $M_1(\bar\varphi, 0)$ and $M_2(-\bar\varphi, 0)$ arise where $\bar\varphi$ is a positive solution of the equation $q(\varphi, s)=0$.
3. In the above problem concerning the oscillations of an aerodynamic pendulum, the loss of stability of the point O occurs simultaneously due to the onset of both antidissipation as well as a repulsive character of the positional force, that is, $a_0=c_0=0$. With a loss of stability of point O, it therefore follows to expect the birth of a stable cycle (or several cycles) in the phase plane to which the self-sustained oscillations of the initial mechanical system correspond.

We next consider in detail the changeovers occurring when the parameter s passes through zero in the case of a combination of two of the above mentioned mechanisms of stability loss. For this purpose, instead of Eqs (1.1) in the neighbourhood of point O, we consider an approximate system obtained using the approximate equations (1.3) with $a_0=c_0=0$ and we call this system (1.1′):

$$\dot\varphi=\omega,\ \dot\omega=-(as+b\varphi^2)\omega-(cs+d\varphi^2)\varphi \tag{1.1′}$$

So, when $s>0$, system (1.1′) has a unique fixed point (1.2), and it is asymptotically stable.

When $s=0$, system (1.1′) only contains non-linear terms. Point O is also a unique quiescent point, and it is stable (although not exponentially stable). The stability of this point was established by constructing the Lyapunov function [6] in investigating system (1.1′) as a special case of the system of more general form.

When $s<0$, together with quiescent point (1.2), a further two quiescent points appear with abscissae

$$\pm\bar\varphi=\pm\sqrt{-cs/d}$$

For convenience in carrying out the parametric analysis, in this case we introduce the notation

$$\varepsilon^2=-as\quad(\varepsilon>0),\quad k=c/a,\quad p=bc/(ad)$$

(the parameters k and p are positive) and make the substitutions

$$\varphi=\varepsilon z\sqrt{c/ad},\ \ \omega=\varepsilon^2\sqrt{c/ad}\,y,\ \ t=\tau/\varepsilon$$

System (1.1) then takes the form

$$\dot z=y,\quad \dot y=k(1-z^2)z+\varepsilon(1-pz^2)y \tag{1.4}$$

that is more convenient for the analysis. The parameter p characterizes the dissipation zone: "antidissipation" of energy occurs in the strip $|z|<z_d=\sqrt{1/p}$ of the phase plane and dissipation outside of it.

The problem is set of describing the cycles existing in system (1.4) for sufficiently small values of ε as a function of the parameter p.

An equation, similar to (1.4), has been considered earlier [3] but the problem of describing the relation between the amplitude of the cycles occurring and the parameter p was not raised.

2. The search for cycles

So, we will consider Eq. (1.1′) when $s<0$ that is equivalent to examining the phase plane of system (1.4).

The system has three quiescent points: $O(0,0)$, $M_1(\bar z, 0)$ and $M_2(-\bar z, 0)$, where $\bar z=1$. Note that, due to the change of coordinates carried out in changing to system (1.4), the quantity $\bar z$ is independent of the parameters ε, p and k.

The coefficients of the characteristic equation $\lambda^2-\varepsilon\lambda-k=0$ corresponding to the point O are negative so that point O is of the saddle type, and it is unstable. It is clear from an analysis of the characteristic equation $\lambda^2-\varepsilon(p-1)\lambda-2k=0$ corresponding to the points $M_{1,2}$