G Model
JAMM-2444; No. of Pages 6
ARTICLE IN PRESS
Journal of Applied Mathematics and Mechanics xxx (2018) xxx–xxx
Contents lists available at ScienceDirect
Journal of Applied Mathematics and Mechanics
journal homepage: www.elsevier.com/locate/jappmathmech

Modelling of a continuously inhomogeneous coating of an elastic sphere by a system of homogeneous elastic layers in the problem of sound scattering☆

L.A. Tolokonnikov

Tula State University, Tula, Russia

ARTICLE INFO

Article history:
Received 20 April 2017
Available online xxx

Keywords:
direction diagram

ABSTRACT

On the basis of analytical solutions of problems on the diffraction of a plane sound wave on a homogeneous elastic sphere with a discretely layered coating and with a continuously-inhomogeneous coating, the direction diagrams of the scattered field are calculated. It is shown that a radially inhomogeneous coating can be modelled by a system of homogeneous elastic layers. For a linear dependence of the inhomogeneity, the number of homogeneous layers in the discretely-layered coating needed to ensure a prescribed accuracy of matching of the direction diagrams of bodies with a discretely layered coating to a continuously-inhomogeneous coating has been determined.

The sound-reflecting characteristics of elastic bodies, including bodies of spherical shape, can be modified using coatings fabricated from functional-gradient materials, whose mechanical properties vary continuously with thickness. A large number of studies have been dedicated to the investigation of the diffraction of sound waves on homogeneous isotropic elastic spherical bodies, for example, Refs 1–3. Sound scattering by continuously-inhomogeneous isotropic elastic spherical layers has been investigated.[4,5] Solutions of problems on scattering of plane, cylindrical, and spherical sound waves by a homogeneous elastic sphere with a radially inhomogeneous coating have been obtained.[6–8] A continuously-inhomogeneous coating has been modelled by a discretely layered one for a thermoelastic plate[9] and for an elastic cylinder.[10]

Below we examine the possibility of modelling a continuously-inhomogeneous coating of an elastic homogeneous sphere by a system of homogeneous elastic layers.

1. Diffraction of a plane sound wave on an elastic sphere with a discretely-layered coating

First let us consider the problem of diffraction of a plane monochromatic sound wave on a homogeneous elastic sphere with a discretely layered coating. Let a homogeneous isotropic elastic sphere of radius r_0, the material of which is characterized by density ρ_0 and elastic constants λ_0 and μ_0, have a coating in the form of a system of N coaxial spherical layers of radii r_j ($j = 1, 2, ..., N$). Each jth homogeneous isotropic elastic layer has density ρ_j and elastic moduli λ_j and μ_j. The body is surrounded by an ideal fluid with equilibrium density ρ and sound velocity in it equal to c.

Let a plane sound wave from the external space be incident onto the sphere. A rectangular Cartesian coordinate system x, y, z and a spherical coordinate system r, θ, ϕ are bound to the sphere. Without loss of generality, we assume that the wave propagates in the z direction. Then the velocity potential of the incident wave is written

$$\Psi_0 = A\exp[i(kr\cos\theta - \omega t)]$$

where A is the wave amplitude, $k = \omega/c$ is the wave number, and ω is the angular velocity. In what follows, we will omit the time factor $e^{-i\omega t}$.

Let us find the wave reflected from the body, and also the displacement fields in the elastic sphere and in the discretely-homogeneous coating. Obviously, by virtue of the axial symmetry of the problem, the excited wave fields will not depend on the φ coordinate.

☆ *Prikl. Mat. Mekh.* Vol. 81, No.6, pp. 699–707, 2017.
E-mail address: tolokonnikovla@mail.ru

https://doi.org/10.1016/j.jappmathmech.2018.03.017

Please cite this article in press as: Tolokonnikov LA. Modelling of a continuously inhomogeneous coating of an elastic sphere by a system of homogeneous elastic layers in the problem of sound scattering. *J Appl Math Mech* (2018), https://doi.org/10.1016/j.jappmathmech.2018.03.017

Propagation of small perturbations in an ideal fluid in the case of steady oscillations is described by the Helmholtz equation[11]

$$\Delta\Psi_{N+1}+k^2\Psi_{N+1}=0$$

where $\Psi_{N+1}=\Psi_0+\Psi_s$ is the velocity potential of the total acoustic wave in the outer region and Ψ_s is the velocity potential of the scattered wave. Here the particles velocity **v** and the acoustic pressure p in the fluid are given by the formulae

$$\mathbf{v}=\operatorname{grad}\Psi_{N+1},\quad p=i\rho\omega\Psi_{N+1}$$

We represent the velocity potential of the incident plane wave in the form[12]

$$\Psi_0(r,\theta)=\sum_{n=0}^{\infty}\gamma_n j_n(kr)P_n(\cos\theta),\quad \gamma_n=Ai^n(2n+1)$$

where $j_n(x)$ is the spherical Bessel function of order n and $P_n(x)$ is the Legendre polynomial of degree n. Taking the radiation conditions at infinity into account,[11] we seek the velocity potential of the scattered wave in the form

$$\Psi_s(r,\theta)=\sum A_n h_n(kr)P_n(\cos\theta) \tag{1.1}$$

where $h_n(x)$ is the spherical Hankel function of the first kind of order n. Here and everywhere below, if not stated otherwise, the sum runs from $n=0$ to $n=\infty$.

Let us now consider the equations describing the propagation of small perturbations in the elastic sphere and in the discretely-layered coating.

We represent the vector of displacements of the particles of the jth elastic isotropic homogeneous elastic layer in the form

$$\mathbf{u}^{(j)}=\operatorname{grad}\Psi^{(j)}+\operatorname{rot}\Phi^{(j)},\quad j=0,1,\ldots,N$$

where $\Psi^{(j)}$ and $\Phi^{(j)}$ are the scalar and vector potentials of the displacements in the jth layer ($j=0$ corresponds to a sphere of radius r_0). In the case of harmonic motion, the displacement potentials are solutions of the Helmholtz wave equations

$$\Delta\Psi^{(j)}+k_l^{(j)2}\Psi^{(j)}=0,\quad \Delta\mathbf{\Phi}^{(j)}+k_\tau^{(j)2}\mathbf{\Phi}^{(j)}=0 \tag{1.2}$$

The wave numbers and velocities of the longitudinal and transverse elastic waves in the jth layer are defined as follows:

$$k_l^{(j)}=\omega/c_l^{(j)},\quad k_\tau^{(j)}=\omega/c_\tau^{(j)}\ \text{и}\ c_l^{(j)}=\sqrt{(\lambda_j+2\mu_j)/\rho_j},\quad c_\tau^{(j)}=\sqrt{\mu_j/\rho_j}$$

Since the problem is axisymmetric, we have

$$\mathbf{\Phi}^{(j)}=\Phi^{(j)}(r,\theta)\mathbf{e}_\varphi$$

where $\mathbf{e}_\varphi$ is the unit vector corresponding to the φ coordinate axis. Thus, vector Eq. (1.2) reduces to one scalar equation in the function $\Phi(r,\theta)$, which in spherical coordinates has the form

$$\Delta\Phi^{(j)}+\left(k_\tau^{(j)2}-\frac{1}{r^2\sin^2\theta}\right)\Phi^{(j)}=0$$

We seek the functions $\Psi^{(j)}$ and $\Phi^{(j)}$ ($j=0,1,\ldots,N$) in the form

$$\Psi^{(j)}(r,\theta)=\sum\left[B_{1n}^{(j)}j_n(k_l^{(j)}r)+B_{2n}^{(j)}h_n(k_l^{(j)}r)\right]P_n(\cos\theta)$$

$$\Phi^{(j)}(r,\theta)=\sum\left[C_{1n}^{(j)}j_n(k_\tau^{(j)}r)+C_{2n}^{(j)}h_n(k_\tau^{(j)}r)\right]\frac{d}{d\theta}P_n(\cos\theta) \tag{1.3}$$

Taking the condition of boundedness for the functions $\Psi^{(0)}$ and $\Phi^{(0)}$ into account, we have

$$B_{2n}^{(0)}=0,\quad C_{2n}^{(0)}=0$$

The coefficients of expansions (1.1) and (1.3) remain to be determined from the boundary conditions. The boundary conditions on the outer surface of the body consist in equality on it of the normal velocities of the particles of the elastic medium and the fluid, equality of the normal stress and the acoustic pressure, and absence of tangential stresses:

$$r=r_N:\quad -i\omega u_r^{(N)}=v_r,\quad \sigma_{rr}^{(N)}=-p,\quad \sigma_{r\theta}^{(N)}=0 \tag{1.4}$$

The components of the displacement vector of the particles and also the normal and tangential stresses

$$r=r_j:\quad u_r^{(j)}=u_r^{(j+1)},\quad u_\theta^{(j)}=u_\theta^{(j+1)},\quad \sigma_{rr}^{(j)}=\sigma_{rr}^{(j+1)},\quad \sigma_{r\theta}^{(j)}=\sigma_{r\theta}^{(j+1)} \tag{1.5}$$

Please cite this article in press as: Tolokonnikov LA. Modelling of a continuously inhomogeneous coating of an elastic sphere by a system of homogeneous elastic layers in the problem of sound scattering. *J Appl Math Mech* (2018), https://doi.org/10.1016/j.jappmathmech.2018.03.017