Journal of Applied Mathematics and Mechanics xxx (2018) xxx–xxx

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The solution of a certain class of dual integral equations with the right-hand side in the form of a Fourier series and its application to the solution of contact problems for inhomogeneous media☆

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ARTICLE INFO

Article history:
Received 19 November 2015
Available online xxx

Keywords:
functionally graded

ABSTRACT

Using the bilateral asymptotic method, a semi-analytical solution of a dual integral equation with its right-hand side in the form of a Fourier series is constructed. This equation arises in the solution of a number of contact problems of elasticity theory for bodies with inhomogeneous coatings. The efficiency of the method is illustrated in the example of the solution of the plane contact problem on bending of a beam lying on a functionally graded strip with arbitrary variation of the elastic moduli with depth. It is assumed that the strip is perfectly bonded to an elastic half-plane. Numerical results are presented for a strip whose Young's modulus varies harmonically with depth. In this case, Young's modulus of the substrate is 100 times greater than at the lower boundary of the coating.

Contact problems for bodies with coatings can be reduced by the method of integral transforms to solution of dual integral equations. Various methods are employed to solve these equations (the regular asymptotic method,[2] the singular asymptotic method,[3] methods of orthogonal polynomials[4] and collocation,[5] etc.), each of them is effective in its own region of values of the characteristic geometrical parameter of the problem. In our work, a dual integral equation is solved by the bilateral asymptotic method,[6] that is under intense development at the present time. The solution is asymptotically exact for small and large values of the characteristic geometrical parameter and has high accuracy for arbitrary intermediate values.

1. Solution of the dual integral equation

Let us consider the dual integral equation (IE)

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{Q(\alpha)}{|\alpha|} L(\alpha\lambda) e^{-ix\alpha}\, d\alpha = 2\pi f(x), \quad |x| \le 1 \\ &\int_{-\infty}^{\infty} Q(\alpha) e^{-ix\alpha}\, d\alpha = 0, \quad |x| > 1 \end{aligned} \tag{1.1}$$

Here $Q(\alpha)$ is the Fourier transform of the function $q(x)$, which is to be determined, λ is the real parameter of the equation, $L(\alpha\lambda)$ is the transformant of the kernel of the dual IE. The solution of a number of plane[7] and antiplane problems in the theory of elasticity, where the function $f(x)$ describes the shape of the surface under the punch (or the shape of the flexible element – the beam), reduces to dual IE (1.1).

Let us consider the case when the function $f(x)$ can be represented in the form of a Fourier series:

$$f(x) = z_0 + \sum_{n=1}^{\infty} z_n \cos(n\pi x) \tag{1.2}$$

☆ *Prikl. Mat. Mekh.* Vol. 81, No.6, pp. 708–716, 2017.

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https://doi.org/10.1016/j.jappmathmech.2018.03.018
0021-8928/© 2018 Published by Elsevier Ltd.

and the unknown function $q(x)$ is even, the as well as the function $L(\alpha\lambda)$. We rewrite dual IE (1.1), taking the even parity of function (1.2) into account, in the form

$$\int_{-\infty}^{\infty}\frac{Q(\alpha)}{|\alpha|}L(\lambda\alpha)\cos(x\alpha)\,d\alpha = 2\pi f(x),\quad |x|\le 1$$
$$\int_{-\infty}^{\infty}Q(\alpha)\cos(x\alpha)\,d\alpha = 0,\qquad |x|>1 \tag{1.3}$$

We assume that the transformant of the kernel possesses the following asymptotic properties:

$$L(\alpha) = A + B|\alpha| + C\alpha^2 + O(|\alpha|^3),\quad \alpha\to 0$$
$$L(\alpha) = 1 + D|\alpha|^{-1} + E\alpha^{-2} + O(|\alpha|^{-3}),\quad \alpha\to\infty \tag{1.4}$$

where A, B, C, and D are constants that depend on the elastic properties of the half-plane. It can be shown that properties (1.4) are fulfilled for a wide class of contact problems for bodies with inhomogeneous coatings.[8] In this case, the transformant can be approximated with high accuracy[9] by expressions of a special kind

$$L(\alpha\lambda)\approx L_N(\alpha\lambda) = \frac{P_1(\alpha^2\lambda^2)}{P_2(\alpha^2\lambda^2)};\quad P_k(\alpha^2\lambda^2) = \prod_{i=1}^{N}(\alpha^2\lambda^2 + a_{ki}^2),\quad k=1,2 \tag{1.5}$$

The constants a_{ki} are, in general, complex parameters of the approximation of the transformant $L(\alpha\lambda)$. Methods of determining these constants were described earlier.[9]

Let us introduce the function

$$d(x) = \frac{1}{2p}\int_{-\infty}^{\infty}\frac{Q(a)}{|\alpha|}\cos(xa)da \tag{1.6}$$

Employing the above-described approach[10] and taking approximation (1.5) into account, we rewrite the first of Eqs (1.3) in operator form

$$P_1(-\lambda^2 D)d(x) = P_2(-\lambda^2 D)f(x),\quad D = \frac{d^2}{dx^2},\quad |x|\le 1 \tag{1.7}$$

Taking the even parity of the function $d(x)$ into account, the solution of differential equation (1.7) has the form

$$d(x) = \sum_{i=1}^{N}\frac{C_i}{a_{1i}\lambda^{-1}}\operatorname{ch}(a_{1i}\lambda^{-1}x) + \sum_{n=0}^{\infty}z_n L_N^{-1}(\lambda\pi n)\cos(\pi n x) \tag{1.8}$$

C_i are unknown constants, to be determined.

Taking expression (1.6) into account we rewrite dual IE (1.3) in the form and differentiating the first equation with respect to x

$$\int_0^{\infty}Q(\alpha)\sin(x\alpha)\,d\alpha = -\pi d'(x),\quad 0\le x\le 1$$
$$\int_0^{\infty}Q(\alpha)\cos(x\alpha)\,d\alpha = 0,\quad x>1 \tag{1.9}$$

The function $d(x)$ is assigned by formula (1.8).

Let us apply the integral operators $\int_0^t \frac{xdx}{\sqrt{t^2-x^2}}$ and $\int_t^{\infty}\frac{xdx}{\sqrt{x^2-t^2}}$ to the first and second of Eqs (1.9), respectively,[11] and carry out the substitution of variables $Q^*(\alpha)=\alpha^{-1}Q(\alpha)$. Finally, after taking the Hankel inverse transform we obtain

$$Q(\alpha) = \tilde{Q}(\alpha) = -2\alpha\int_0^1 J_1(\alpha t)\,dt\int_0^t\frac{xd'(x)dx}{\sqrt{t^2-x^2}}$$

Since we used the operation of differentiation in transforming from dual IE (1.3) to Eqs (1.9), it is necessary, in addition, to take into account[11] the solution of the homogeneous equation (1.9). It is easy to show that this solution has the form $Q(\alpha)=FJ_0(\alpha)$, where F is an unknown constant. Finally, we obtain the equality

$$Q(\alpha) = FJ_0(\alpha) + \tilde{Q}(\alpha) \tag{1.10}$$

Inverting the Fourier transform, we obtain an analytical representation for the function $q(x)$

$$q(x) = \frac{F}{\pi\sqrt{1-x^2}} + \sum_{j=1}^{N}C_j\Psi(a_{1j}\lambda^{-1},x) + \sum_{n=1}^{\infty}\frac{z_n\pi n i}{L_N(\lambda\pi n)}\Psi(i\pi n,x) \tag{1.11}$$

$$\Psi(A,x) = \frac{I_1(A)}{\sqrt{1-x^2}} - A\int_x^1[I_0(A)\alpha\operatorname{ch}A(\alpha-x) - I_1(A)\operatorname{sh}A(\alpha-x)]\frac{d\alpha}{\sqrt{1-\alpha^2}}$$

J_0, J_1 and I_0, I_1 are ordinary and modified Bessel functions of the first kind.

Please cite this article in press as: Aizikovich SM, et al. The solution of a certain class of dual integral equations with the right-hand side in the form of a Fourier series and its application to the solution of contact problems for inhomogeneous media. *J Appl Math Mech* (2018), https://doi.org/10.1016/j.jappmathmech.2018.03.018