Journal of Applied Mathematics and Mechanics xxx (2018) xxx–xxx

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Contact problem for a hollow cylinder☆

D.A. Pozharskii

Don State Technical University, Rostov-on-Don, Russia

ARTICLE INFO

Article history:
Received 6 December 2016
Available online xxx

Keywords:
sleeve asymptotics

ABSTRACT

The axisymmetric contact problem of the interaction of a rigid annular sleeve with an infinite hollow elastic cylinder with an arbitrary wall thickness, which is subjected to the action of a constant internal pressure, is investigated. When the solution of Lame's problem for a hollow cylinder and the integral transformation method are used, the contact problem is reduced to an integral equation with a difference kernel relative to the unknown pressure in the contact area. To solve this equation in the case of relatively wide sleeves, a modification of the singular asymptotic method based on complication of the approximating function for the symbol function of the kernel when the cylinder walls are made thinner is proposed. Calculations are performed for a broad range of variation of the relative thickness of the cylinder walls with approach to values that are characteristic of the theory of cylindrical shells, in which the shell thickness usually amounts to no more than 2% of the radius of the middle surface. ©2017

Asymptotic methods were previously used to study similar contact problems for an infinite continuous cylinder and a space with a cylindrical cavity,[1–4] whose integral equations are limit special cases of the equation obtained in the present work. Problems concerning the interaction of several pointed punches with a cylindrical shell[5] and the contact of a cylindrical shell[6] and an elastic ring[7] with a continuous cylinder have been studied. The contact problem of the wear of a hollow cylinder has been investigated.[8]

In the case considered below, the asymptotics of the symbol function of the kernel of the integral equation at zero and infinity are more complicated than in the case of a continuous cylinder or a space with a cylindrical cavity.[3] Nevertheless, the identical nature of the structure of the principal terms of the asymptotics for a continuous cylinder and a hollow cylinder makes it possible to complicate the known approximation[3] for the purpose of taking into account the variation of the cylinder wall thickness. The problem is timely for analysing the stress redistribution in pipelines when possible leaks are eliminated.

1. Integral equation of the problem

Under the conditions of axial symmetry, using the cylindrical coordinates r, z, we will consider a hollow elastic cylinder $\{\rho_1 \le r \le \rho, |z| < \infty\}$ with the shear modulus G and Poisson's ratio ν under the action of the uniform internal pressure p_0. A rigid annular sleeve of width $2a$ is placed on the outside of the cylinder with the traction δ. The remainder of the outer surface is not loaded, and the tangential stresses on the entire surface are equal to zero. It is required to determine the contact pressure $\sigma_r(\rho, z) = -q(z)$ in the contact area $|z| \le a$.

By virtue of the principle of superposition of solutions, we write the contact condition in the form

$$u_0(\rho, z) + u_1(\rho, z) = -\delta, \; |z| \le a \tag{1.1}$$

where $u_0(\rho, z)$ is the normal displacement of the outer surface determined using the integral Fourier transform when $p_0 = 0$ and $q(z) \neq 0$, and $u_1(\rho, z)$ is the displacement obtained when $q(z) = 0$ from the solution of Lame's problem for a cylinder under the action of the internal pressure $\sigma_r(\rho, z) = -p_0$ (Ref. 9):

$$u_1(\rho, z) = \frac{p_0 \rho k^2}{\theta(1-\nu^2)(1-k^2)}, \quad \theta = \frac{G}{1-\nu}, \quad k = \frac{\rho_1}{\rho} \tag{1.2}$$

☆ *Prikl. Mat. Mekh.* Vol. 81, No. 6, pp. 727–733, 2017.
E-mail address: pozharda@rambler.ru

https://doi.org/10.1016/j.jappmathmech.2018.03.020

Solving the auxiliary boundary value problem for $p_0 = 0$ and taking into account expressions (1.1) and (1.2), we arrive at the integral equation for the function $q(z)$

$$\int_{-a}^{a} q(\zeta)K_*(z-\zeta)\,d\zeta = \pi\left(\frac{p_0k^2}{(1-\nu^2)(1-k^2)} + \frac{\theta\delta}{\rho}\right), \quad |z| \le a \tag{1.3}$$

with the kernel

$$K_*(t) = \int_0^\infty L_0(s)\cos(st)\,ds, \quad L_0(s) = \frac{L_1(s)}{2(1-\nu)L_2(s)} \tag{1.4}$$

where

$$L_1(s) = (I_1K_1^* - K_1I_1^*)^2 - 4F_1^*I_1K_1 - 2F_2^*K_1^2 - 2F_3^*I_1^2$$

$$L_2(s) = -(I_1K_1^* - K_1I_1^*)^2 + 4F_1^*(I_1K_1 - I_1^*K_1^*) + 2F_2^*(K_1^2 - K_1^{*2}) +$$
$$+ 2F_3^*(I_1^2 - I_1^{*2}) - 4F_1^{*2} + 4F_2^*F_3^*$$

$$F_1^* = \frac{s^2}{4(1-\nu)}[\rho_1^2(I_0^*K_0^* + I_1^*K_1^*) - \rho^2(I_0K_0 + I_1K_1)]$$

$$F_2^* = \frac{s^2}{4(1-\nu)}[\rho_1^2(I_0^{*2} - I_1^{*2}) - \rho^2(I_0^2 - I_1^2)]$$

$$F_3^* = \frac{s^2}{4(1-\nu)}[\rho_1^2(K_0^{*2} - K_1^{*2}) - \rho^2(K_0^2 - K_1^2)]$$

$$I_n = I_n(\rho s), \quad K_n = K_n(\rho s), \quad I_n^* = I_n(\rho_1 s), \quad K_n^* = K_n(\rho_1 s); \quad n = 0, 1 \tag{1.5}$$

Here $I_n(u)$ and $K_n(u)$ are modified Bessel functions.[10]

By virtue of the asymptotic behaviour of Bessel functions at zero and infinity,[10] when $\rho = \text{const}$ and $\rho_1 \to 0$, we have

$$L_0(s) \to \frac{I_1^2}{s^2\rho^2(I_0^2 - I_1^2) - 2(1-\nu)I_1^2}$$

which coincides at the limit with the known symbol function for a continuous cylinder (Ref. 2, Eq. (6.53)). When $\rho_1 = \text{const}$ and $\rho \to \infty$, we have

$$L_0(s) \to \frac{K_1^{*2}}{s^2\rho_1^2(K_0^{*2} - K_1^{*2}) - 2(1-\nu)K_1^{*2}}$$

which is identical at the limit to the known symbol function for a space with a cylindrical cavity (Ref. 2, Eq. (6.54)) apart from the sign. In this case, according to the superposition principle, the load applied at infinity can be transferred onto the cavity surface (with the opposite sign).

After introducing the dimensionless quantities

$$x = \frac{z}{a}, \quad u = \rho s, \quad \lambda = \frac{\rho}{a}, \quad \varphi(x) = \frac{q(z)}{\theta}, \quad f = \frac{p_0\lambda k^2}{\theta(1-\nu^2)(1-k^2)} + \frac{\delta}{a} \tag{1.6}$$

into formulae (1.3)–(1.5), we obtain the following integral equation for the function $\varphi(x)$:

$$\int_{-1}^{1} \varphi(\xi)K\left(\frac{x-\xi}{\lambda}\right)d\xi = \pi f, \quad |x| \le 1; \quad K(t) = \int_0^\infty L(u)\cos ut\,du, \quad L(u) = \frac{L_1(u)}{2(1-\nu)L_2(u)} \tag{1.7}$$

The functions $L_1(u)$ and $L_2(u)$ are found from formulae (1.5), in which s, ρ_1 and ρ should be replaced by u, k and 1, respectively.

The dimensionless parameters $k \in [0,1)$ and λ, which were introduced into formulae (1.2) and (1.6), characterize the relative thickness of the cylinder walls and the relative width of the sleeve, respectively.

After the replacement indicated above, the symbol function $L(u)$ defined by formulae (1.7), (1.4) and (1.5) has the following asymptotic behaviour at zero and infinity:

$$L(u) = A + o(1), \quad u \to +0; \quad A = \frac{1}{2(1+\nu)} + \frac{k^2}{(1-\nu^2)(1-k^2)}$$
$$L(u) = \frac{1}{u} + \frac{c_1}{u^2} + o(u^{-2}), \quad u \to +\infty; \quad c_1 = 1 - 2\nu \tag{1.8}$$

When $k = 0$, the parameters in formulae (1.8) are identical to the known parameters for a continuous cylinder.[3] When the cylinder walls become thinner ($k \to 1$), we observe a significant increase in the parameter $A = L(0)$ and expansion of the range of values of the symbol function $L(u) \in (0, A]$, which complicates its approximation for obtaining an explicit asymptotic solution of the problem.