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The angular motion of oceanic vortex formations☆

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ABSTRACT

Within the framework of the rigid body hypothesis, the influence of external torques acting on a rotating water lens in a stratified ocean is examined and a hypothesis about the angular motion of objects of such kind is constructed. The structure of the external torques acting on the lens is investigated and their magnitudes and influence on the overall picture of the motion of the lens about its centre of mass are estimated. It is shown that the hydrostatic buoyancy torque is the most important of such torques, it being orders of magnitude greater than the Coriolis torque and the torque due to virtual masses, and also the gravitational torque and other torques. The friction torque can promote stabilization of the angular motion and lead to the appearance of a steady regime. The results obtained are in agreement with the observed motion of oceanic formations.

1. Statement of the problem

In the ocean, vortex formations (lenses) have been observed, moving in the depth of the ocean and almost unnoticeable at the surface. They move, deform, and are washed by flows, in the course of which they rotate as a unified rigid body with angular velocity vector directed almost along the local vertical to the Earth.[1–5] Employing methods of theoretical mechanics, it is possible to attempt an explanation of the motion of such a lens as a rotating rigid body (the solid body hypothesis) moving in an ideal incompressible fluid on a rotating sphere, in a non-inertial coordinate system bound to the sphere. A number of papers (e.g., Refs 2 and 5) have examined the rotational motion of lenses, but they contain only fragmentary information about the character of the torques acting on these vortex formations.

The main reasons for the appearance of angular torques are due to three factors: rotation of the Earth, stratification of the ocean, and hydrodynamic effects associated with the appearance of perturbations in the surrounding aqueous medium. It goes without saying that in a rigorous approach it would be necessary to account for their combined effect on the basis of some combined system of dynamical conditions. A method of taking them into account separately can only be approximate, just like the rigid body hypothesis of the lenses, both of which clearly require further analysis.

1.1. Torques caused by the rotation of the Earth

We introduce two reference frames (coordinate systems) with origin at the centre O of the spherical Earth (left side of the figure): the absolute system $OXYZ$ and the system $OX_1Y_1Z_1$ bound to the Earth rotating with constant angular velocity Ω, and also three systems with origin at the centre of mass of the lens C (right side of the figure): the König system $CXYZ$, the system $CX_1Y_1Z_1$, and the system $Cxyz$ bound to the principal central axes of the lens.

In the system $CXYZ$, the theorem on the variation of the kinetic angular momentum vector $\mathbf{L}$ of the dynamical system for angular motion of the lens

$$d\mathbf{L}/dt = \mathbf{M}$$

is valid, where M is the of external torque about the point C.

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It is more convenient to use a similar relation written in the coordinate system $CX_1Y_1Z_1$ bound to the Earth, which requires that we take account of the inertial torques:

$$d\mathbf{L}/dt = \mathbf{M} + \sum_i [\mathbf{r}_i \times dm_i(-\mathbf{j}_n)] + \sum_i [\mathbf{r}_i \times dm_i(-\mathbf{j}_c)] \equiv \mathbf{M} + \mathbf{M}_c + \mathbf{M}_n \tag{1.1}$$

Here dm_1 is the element of mass, $\mathbf{r}_1$ is the radius vector of the ith point mass of the lens relative to the point C, $\mathbf{j}_n$ and $\mathbf{j}_c$ are vectors of accelerations of the ith point mass of the lens: the transport acceleration (rotation of the lens with angular velocity $\mathbf{\Omega}$ of rotation of the Earth with respect to the system of axes $CXYZ$) and the Coriolis acceleration (motion of points of the lens relative to the fixed axes $CX_1Y_1Z_1$ due to rotation of the axes $Cxyz$ with angular velocity $\boldsymbol{\omega}$). In the case of a symmetric solid body, the Coriolis and transport torques are given by the formulae[6]

$$\mathbf{M}_c = C[\omega \times \Omega] + 2(A - C)[\omega \times \mathbf{z}](\Omega, \mathbf{z}), \quad \mathbf{M}_n = (C - A)(\mathbf{z}, \Omega)[\mathbf{z} \times \Omega] \tag{1.2}$$

Here $\mathbf{z}$ is the unit vector of the symmetry axis, and A and C are the principal central torques of inertia of the lens.

The Coriolis forces of inertia act only on point masses of the lens, while the transport forces of inertia, static by their nature, act to an equal degree on the particles of the unperturbed surrounding ocean. Since the ocean is in equilibrium with respect to the Earth, additional torques acting on the lens compared with the action of the particles of the ocean are also of interest. Therefore it makes sense to take into account the transport torque (the second of equalities (1.2)) with a small coefficient of compensation Δ (the compensation hypothesis), and thus

$$\mathbf{M}_n = \Delta(C - A)(\mathbf{z}, \Omega)[\mathbf{z} \times \Omega] \tag{1.3}$$

As an estimate of the coefficient Δ it is possible to consider, for example, the relative difference of the masses of the lens and the water displaced by it.

1.2. Torque due to the reaction of the medium

The motion of the lens relative to the Earth entrains into motion the water layers adjacent to the lens. An additional torque of the reaction of the medium arises, as does also a hydrodynamic torque. The torque of the reaction of the medium is described in the model of an ideal fluid by a formula[7] containing $\mathbf{V}(v_1, v_2, v_3)$ and $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3)$, the vectors of the linear and angular velocity of the centre of mass of the lens in the coordinate system $CX_1Y_1Z_1$ bound to the rotating Earth, in the projections onto the principal axes of the lens $Cxyz$. This formula also contains some coefficients λ_{ij} $(i, j = 1, ..., 6)$, characterizing the magnitudes of the virtual masses; their exact values are unknown and require estimates based on the character of the fluid motion (see, for example, Ref. 7). In the case of an axisymmetric body, this torque can be represented by two terms, one of which has the form

$$\mathbf{M}_m = \lambda(\mathbf{z}, \mathbf{V})[\mathbf{z} \times \mathbf{V}] \tag{1.4}$$

(λ is some coefficient) and can be considered as one of the terms of the principal torque $\mathbf{M}$ on the right-hand side of relation (1.1), and the other, preliminarily represented in Euler form, on the left-hand side of the same relation

$$d\mathbf{L}/dt + [\omega \times \mathbf{L}] = \mathbf{M} \tag{1.5}$$

We will now understand the vector $\mathbf{L}$ on the left-hand side of Eq. (1.5) as the kinetic torque of a body consisting of the point masses of the lens in the presence of the reaction of the medium, and we write it in projections onto the axes of the moving coordinate system $Cxyz$.

The moments of inertia of such a body are A^* $(A^* > A)$ and C; for an oblate spheroid $\lambda > 0$.

1.3. The buoyancy torque

Since the medium surrounding the lens is stratified, besides the dynamic torques, the hydrostatic torques acts on it. For low velocities of the lens motion, this is one of the main acting torques.

Let us consider a lens in the form of a spheroid of uniform density ρ with its centre at the point C, with semi-axes $a = b$ and c, where the distribution of density of the surrounding medium obeys the linear law $\rho(h) = \rho_0 - q(h - h_0)$, where ρ_0, h_0 $(=0)$, and q are constants, and the value of q has the sense of the density gradient of the medium $d\rho/dh$. In the static case, the pressure p is given by the formulae

$$\partial p/\partial x = 0, \quad \partial p/\partial y = 0, \quad \partial p/\partial h = -g\rho(h)$$

where g is the acceleration due to gravity. Following the usual approach[7] and writing the expression for the principal torques $\mathbf{M}$ due to the buoyancy (hydrostatic) forces exerted by the surrounding medium, to start with, in the form of a surface integral, and then, utilizing Gauss' formula, in the form of a volume integral, we obtain the following expression for the buoyancy torque:

$$\mathbf{M}_a = (A - C)N^2(\mathbf{n}, \mathbf{z})[\mathbf{n} \times \mathbf{z}] = (a^2 - c^2)mN^2(\mathbf{n}, \mathbf{z})[\mathbf{z} \times \mathbf{n}]/5 \equiv \mu(\mathbf{n}, \mathbf{z})[\mathbf{z} \times \mathbf{n}] \tag{1.6}$$

where A and C are the torques of inertia of the body filled with a homogeneous fluid with density ρ_0, N is the Väisäla–Brunt frequency of the ocean at the level of the centre of mass of the lens, $\mathbf{n}$ is the unit vector of the local upward vertical at the centre of mass of the lens, i.e., at the point C, m is the mass of the homogeneous fluid of density ρ_0 displaced by the lens; in principle, the values of ρ and ρ_0 can be different if the lens has not reached its equilibrium level in the ocean. However, in reality the differences are small; therefore, to estimate the values of A and C we can take the corresponding torques of inertia of the lens itself. In the case of a lens of spheroidal shape, we have

$$A = m(b^2 + c^2)/5, \quad C = m(a^2 + b^2)/5$$