ARTICLE IN PRESS

Probabilistic Engineering Mechanics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Exact stochastic analysis of coupled bending–torsion beams with in-span supports and masses

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A R T I C L E I N F O

Keywords:
Random loads
Coupled bending–torsional vibrations
Generalized functions
Elastic supports
Attached masses

A B S T R A C T

The stochastic response of a coupled bending–torsion beam, carrying an arbitrary number of supports/masses, is investigated. Using the theory of generalized functions in conjunction with the Euler–St. Venant coupled bending–torsion beam theory, exact analytical solutions under stationary inputs are obtained based on frequency response functions derived by two different closed-form expressions. The analytical solutions are obtained for all response variables, considering any number of supports/masses along the beam and arbitrary spatial load distributions. Two numerical examples are reported.

1. Introduction

The response of beams to random loads has been widely investigated in literature, and some examples can be found in Refs. [1–5]. Most works concerned bending vibration analysis based on Euler–Bernoulli or Timoshenko beam theory, under the fundamental assumption that the beam cross section is doubly-symmetric, so that the shear center (SC) coincides with the mass center (MC). However, beams with mono-symmetric cross sections are frequently employed in different engineering applications such as wings, turbine blades and propellers. Because, in this case, the SC does not coincide with the MC, the beam dynamics exhibits coupled bending–torsion phenomena. Indeed, bending-induced inertial forces, which arise along the mass axis (i.e. the locus of the MCs of the beam cross sections), are eccentric with respect to the elastic axis (i.e. the locus of the SCs of the beam cross sections) and, as a result, bending is inherently coupled with twisting. The coupling effects have been described by the so-called Euler–St. Venant coupled bending–torsion beam theory [6–10], i.e. neglecting warping effects and not considering rotatory inertia and shear deformation of the beam. In a more general case, these effects were considered in other studies [11,12].

As for forced vibration analysis, using the normal mode method Eslimy et al. [13,14] obtained the response of a coupled bending–torsion beam when subjected to deterministic as well as stationary, Gaussian loads. Specifically, they applied the method to a cantilever aircraft wing for which there is a substantial coupling between bending and torsional vibrations.

It must be noticed, however, that all works in Ref. [1] through Ref. [14] generally addressed uniform beams, with no attachments or in-span supports. The latter are of great interest for engineering applications but have rarely been considered in studies on coupled bending–torsion phenomena [15,16].

This paper proposes an exact method for beams with mono-symmetric cross section, carrying an arbitrary number of elastic supports and attached masses, subjected to stationary stochastic loads. The Euler–St. Venant bending–torsion theory is considered as in Refs. [6–10], in conjunction with the theory of generalized functions [17–19] to handle the discontinuities of the response variables at the application points of supports and masses. Exact analytical solutions for the power spectral density of the response are built based on frequency response functions of the beam obtained, in this paper, by two different closed-form expressions. The key step to build the frequency response functions is a novel analytical expression derived for the response of the beam without supports/masses, subjected to arbitrarily-placed harmonic unit force and unit twisting moment. Two numerical examples will be carried out, including a comparison with results obtained by Euler–Bernoulli beam theory, i.e. neglecting bending–torsion coupling effects.

2. Problem statement

Fig. 1 shows the case under study, i.e. a uniform straight beam of length L, referred to a right handed coordinate system (Oxyz), carrying an arbitrary number N of elastic supports and attached masses, and subjected to a transverse distributed load. The beam cross section is assumed to be mono-symmetric, being x its symmetry axis. The loci of the SCs and MCs of the beam cross sections are respectively the elastic

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http://dx.doi.org/10.1016/j.probengmech.2017.07.002
Received 31 May 2017; Accepted 12 July 2017
Available online xxxx
0266-8920/

Please cite this article in press as: A. Burlon, et al., Exact stochastic analysis of coupled bending–torsion beams with in-span supports and masses, Probabilistic Engineering Mechanics (2017), http://dx.doi.org/10.1016/j.probengmech.2017.07.002.

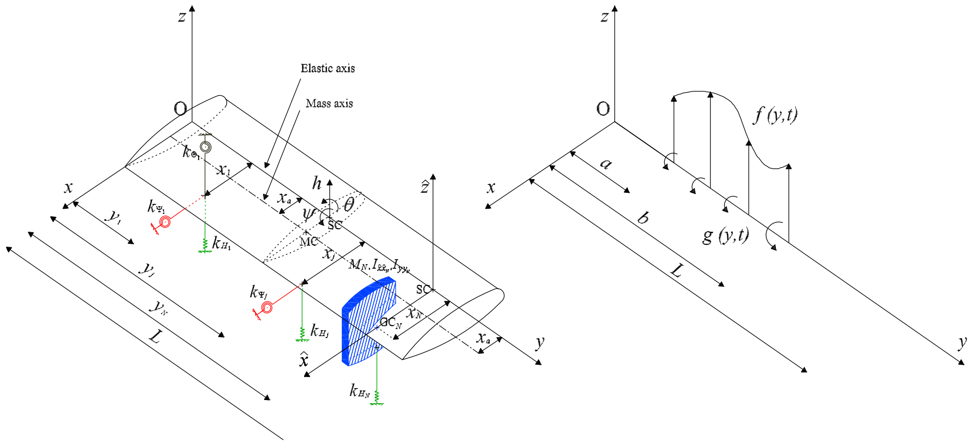

Fig. 1. Beam with mono-symmetric cross section carrying an arbitrary number of elastic supports and attached masses, subjected to distributed loads.

axis and the mass axis; the first coincides with the y-axis, while the latter is at distance x_a from the y-axis. The bending deflection in z-direction, the bending rotation about the x-axis and the torsional rotation about the y-axis are denoted respectively by $h(y,t), \theta(y,t)$ and $\psi(y,t)$.

It is assumed that the transverse distributed load acts in a plane parallel to the $y-z$ plane, at distance x_c from the y-axis (elastic axis); it can be separated into a transverse force $f(y,t)$ applied along the y-axis and a distributed twisting moment $g(y,t)$ about the y-axis, as shown in Fig. 1. The elastic supports and attached masses are applied at y_j, with $0 < y_1 < \cdots < y_j < \cdots < y_N < L$ and stiffness parameters of the jth support and properties of the jth mass are denoted as follows:

- k_{H_j} for translational support, k_{Ψ_j} for torsional–rotational support and k_{Θ_j} for bending-rotational support.
- M is the mass, $I_{\hat{x}\hat{x}_j}$ and I_{yy_j} are the components of the mass inertia tensor about axes $\hat{x}$ and y in Fig. 1.

Some assumptions are made on the attachments:

1. The gravity center (GCj) of the jth attached mass is at distance $z_j = 0$ from the y-axis (elastic axis).
2. The principal axes of the jth mass are parallel to those of the beam cross section.
3. The jth support and gravity center GC$_j$ of the jth mass are both applied at distance x_j from the elastic axis.

The first two assumptions (1), (2) ensure that twisting is coupled with bending in z-direction only, as in Refs . [7–14]. Assumption (3) is made for simplicity but can easily removed, at the expense of minor changes in the derived solutions.

Pursuing the primary purpose of obtaining the frequency response functions for computing the stochastic response, assume that the beam carrying supports/masses, as shown in Fig. 1, is loaded by a harmonic distributed force $f(y,t) = f(y)e^{i\omega t}$, and a harmonic distributed twisting moment $g(y,t) = g(y)e^{i\omega t} = f(y)x_c e^{i\omega t}$ on the interval (a,b), where $0 \le a \le b \le L$ and ω is the frequency. For generality, it is assumed that $f(y)$ is a polynomial function. Let $h(y,\omega,t) = H(y,\omega)e^{i\omega t}$ and $\psi(y,\omega,t) = \Psi(y,\omega)e^{i\omega t}$ be the steady-state response variables. By using the Euler–St. Venant coupled bending–torsion theory and making use of the generalized functions (for details see Appendix A), the following steady-state coupled equations of motion are derived (frequency dependence of response variables is omitted for brevity):

$$EI\frac{\bar{d}^4H}{dy^4} - (m\omega^2 - c_h i\omega)H + (mx_a\omega^2 - c_h x_a i\omega)\Psi - \sum_{j=1}^{N} P_j\delta(y-y_j) + \sum_{j=1}^{N} Mf_j\delta^{(1)}(y-y_j) - f(y) = 0 \quad (1)$$

$$GJ\frac{\bar{d}^2\Psi}{dy^2} - (m\omega^2 x_a - c_h i\omega x_a)H + (I_\alpha\omega^2 - c_\psi i\omega)\Psi - \sum_{j=1}^{N} P_j x_j\delta(y-y_j) + \sum_{j=1}^{N} Mt_j\delta(y-y_j) + g(y) = 0 \quad (2)$$

where bar means generalized derivative, EI and GJ are respectively bending and torsional rigidities, m is the mass per unit length, I_α is the polar moment of inertia per unit length about the elastic axis, while c_h and c_ψ are viscous damping coefficient per unit length respectively in bending and torsion, to be assigned so that damping is proportional [14]. In Eqs. (1)–(2), $\delta^{(k)}(y-y_j)$ is the kth formal derivative of Dirac's delta $\delta(y-y_j)$ [17], while P_j, Mt_j and Mf_j are concentrated force, twisting moment and bending moment associated with supports and mass at y_j, given as:

$$P_j(\omega) = -\kappa_{P_j}(\omega)\left[H(y_j,\omega) - x_j\Psi(y_j,\omega)\right] \quad (3)$$

$$Mt_j(\omega) = -\kappa_{T_j}(\omega)\Psi(y_j,\omega) \quad (4)$$

$$Mf_j(\omega) = -\kappa_{M_j}(\omega)\Theta(y_j,\omega) \quad (5)$$

being $H(y_j,\omega)$, $\Psi(y_j,\omega)$ and $\Theta(y_j,\omega)$ the deflection, torsional and bending rotation at $y = y_j$, while $\kappa_{P_j}(\omega)$, $\kappa_{T_j}(\omega)$, $\kappa_{M_j}(\omega)$ are frequency-dependent terms given as

$$\kappa_{P_j}(\omega) = k_{H_j} - M_j\omega^2 \quad (6)$$

$$\kappa_{T_j}(\omega) = k_{\Psi_j} - (I_{yy_j} - M_j x_j^2)\omega^2 \quad (7)$$

$$\kappa_{M_j}(\omega) = k_{\Theta_j} - I_{\hat{x}\hat{x}_j}\omega^2 \quad (8)$$

It is noticed that the governing equations (1)–(2) of the beam with in-span supports and attached masses, depicted in Fig. 1, have been written for a general case of supports and masses occurring simultaneously at every location y_j. It will be shown later in the paper that removing this assumption produces simple changes in the derived solutions.

Eqs. (1)–(2) can be combined, by eliminating either H or Ψ, to obtain two 6-th order differential equations for deflection and torsional rotation:

$$\alpha\frac{\bar{d}^6H}{dy^6} + \beta\frac{\bar{d}^4H}{dy^4} - \gamma\frac{\bar{d}^2H}{dy^2} + \eta H - \frac{I_a\omega^2 - c_\psi i\omega}{mx_a\omega^2 - c_h i\omega x_a}f(y) - \frac{GJ}{mx_a\omega^2 - c_h i\omega x_a}f^{[2]}(y) - g(y) + R_{H_{ext}}(y) = 0 \quad (9)$$

$$\alpha\frac{\bar{d}^6\Psi}{dy^6} + \beta\frac{\bar{d}^4\Psi}{dy^4} - \gamma\frac{\bar{d}^2\Psi}{dy^2} + \eta\Psi - f(y) + \frac{EI}{mx_a\omega^2 - c_h i\omega x_a}g^{[4]}(y) - \frac{m\omega^2 - c_h i\omega}{mx_a\omega^2 - c_h i\omega x_a}g(y) + R_{\Psi_{ext}}(y) = 0 \quad (10)$$

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