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Multivariate distribution model for stress variability characterisation

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ABSTRACT

In situ stress is an important parameter in rock mechanics, but localised measurements of stress often display significant variability. For improved understanding of *in situ* matrices that satisfy both Eq. stress it is important that this variability be correctly characterised, and for this a robust statistical distribution model – one that is faithful to the tensorial nature of stress – is essential. Currently, variability in stress measurements is customarily characterised using separate scalar and vector distributions for principal stress magnitude and orientation respectively. These customary scalar/vector approaches, which violate the tensorial nature of stress, together with other quasi-tensorial applications found in the literature that consider the tensor components as statistically independent variables, may yield biased results. To overcome this, we propose using a multivariate distribution model of distinct tensor components to characterise the variability of stress tensors referred to a common Cartesian coordinate system. We discuss why stress tensor variability can be sufficiently and appropriately characterised by its distinct tensor components in a multivariate manner, and demonstrate that the proposed statistical model gives consistent results under coordinate system transformation. Transformational invariance of the probability density function (PDF) is also demonstrated, and shows that stress state probability is independent of the coordinate system. Thus, stress variability can be characterised in any convenient coordinate system. Finally, actual *in situ* stress results are used to confirm the multivariate characteristics of stress data and the derived properties of the proposed multivariate distribution model, as well as to demonstrate how the quasi-tensorial approach may give biased results. The proposed statistical distribution model not only provides a robust approach to characterising the variability of stress in fractured rock mass, but is also expected to be useful in risk- and reliability-based rock engineering design.

1. Introduction

In situ stress is an important parameter in many aspects of rock mechanics, including hydraulic fracturing propagation, rock mass permeability analysis and earthquake potential evaluation.[1–5] Because of the inherent complexity of fractured rock masses in terms of varying rock properties and the presence of discontinuities, *in situ* stress in rock often displays significant variability.[4,6–8] Also, with the increasingly widespread application of probabilistic or reliability-based design in rock engineering,[9–14] robustly incorporating stress variability in these analyses is becoming a necessity and for this a statistical distribution model is a prerequisite. However, stress is a second order tensor, and it appears that a robust statistical distribution model that characterises the variability of stress data – one that is faithful to the tensorial nature of stress – is not available. To address this deficiency, and to assist probabilistic-related analyses that need to consider the inherent variability of *in situ* stress, we propose using a multivariate distribution model for stress variability characterisation.

Currently in rock mechanics, stress magnitude and orientation are customarily processed separately (e.g. Fig. 1). This effectively decomposes the stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, to which non-tensor related approaches such as classical statistics[15] and directional statistics,[16] respectively, are applied.[6,9,17–32] However, such applications imply a statistical distribution model that is an *ad hoc* combination of distributions of scalars and vectors, and therefore in general are erroneously applying statistical tools to process data that are referred to different geometrical bases. They thus violate the tensorial nature of stress, and as a result may yield biased results.[33–40] Additionally, these non-tensor related statistical models render it difficult to incorporate stress variability into reliability-based geotechnical engineering design codes such as Eurocode 7.[41]

As an alternative to the separate analysis of principal stress magnitude and orientation, and to be faithful to the tensorial nature of stress, analyses of stress variability should be conducted on the basis of stress tensors referred to a common Cartesian coordinate system.[34,38–40]

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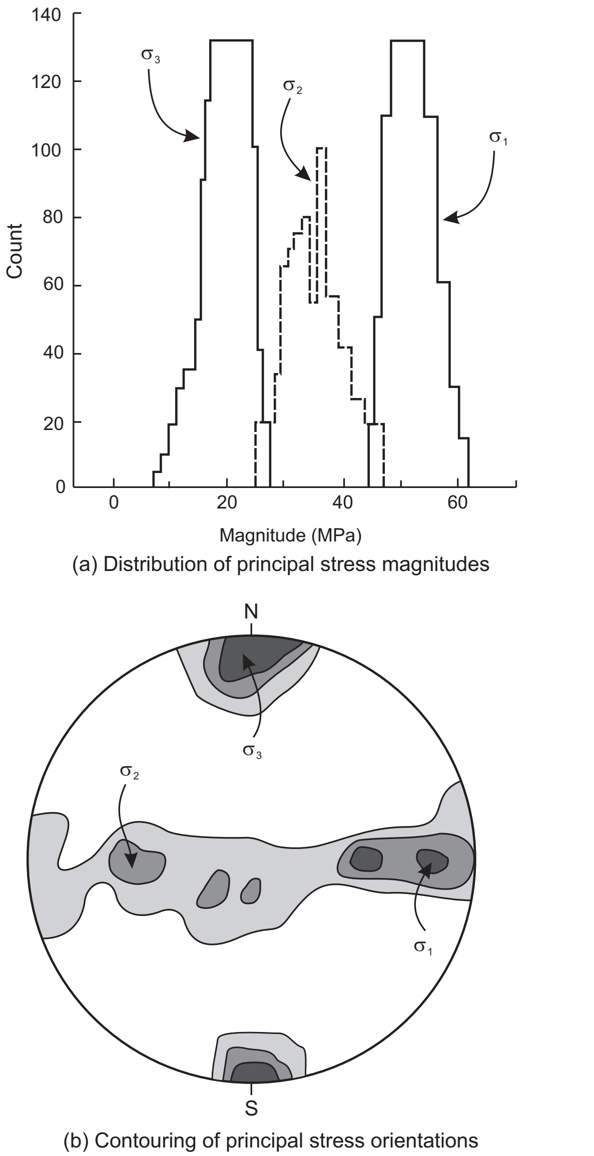

Fig. 1. Customary analyses of stress processes principal stress magnitude and orientation separately using classical statistics and directional statistics, respectively (after Brady & Brown[31]).

Several researchers have followed this technique when calculating the mean[36,42–45] and variance[36,43,45,46] of stress tensors, as well as for random stress tensor generations.[43,46,47] However, in this previous work the stress tensor components are treated as statistically independent variables, which implies an underlying statistical distribution model that is a combination of several independent univariate distributions in which the effect of correlation between tensor components is ignored.[39] As with earlier non-tensorial customary scalar/vector approaches, these quasi-tensorial applications may also yield unreasonable results. We have previously discussed the inappropriateness of customary scalar/vector and quasi-tensorial approaches for stress variability characterisation.[38–40]

In order to improve on these oversimplified statistical distribution models, some work has attempted to apply multivariate statistics to analyses of stress variability.[36,48,49] However, multivariate statistics is generally only suitable for vector data, not tensors.[50] We therefore suggest that these previous applications of multivariate techniques have taken place in an empirical setting, in that the applicability of multivariate statistics to stress variability analysis has not been formally demonstrated. Thus, to date there seems to have been no mathematically rigorous proposal for, and systematic analysis of, a statistical distribution model for stress variability characterisation in rock mechanics. The principal aim of this article is to provide this formal framework.

Stress tensors, which are 2×2 or 3×3 symmetric matrices, together with other matrix-valued quantities, play a pivotal role in many subjects such as solid mechanics, physics, earth science, medical imaging and economics.[51] To explicitly account for the inherent variability of such matrix-valued quantities, matrix variate statistics – as a generalisation of multivariate statistics – has been developed.[51] Although this has been demonstrated to be appropriate for stress variability analysis,[52] application of it is not straightforward and some essential components remain to be developed.[53,54] Fortunately, the statistical equivalence between matrix variate and multivariate statistics implies that, for stress variability analysis, multivariate statistics can be used in certain circumstances as an easily-applicable alternative to matrix variate statistics. Indeed, matrix variate statistics and multivariate statistics are occasionally used interchangeably.[51,54–56]

Among many statistical distributions, the normal distribution is particularly important as physical observations are often seen to be approximately normally distributed.[51] Thus, to provide an easily applicable approach, and as an extension of our previous work,[33,52] here, based on matrix variate statistics and using the normal distribution as an example, a multivariate distribution model for characterising the variability of stress tensors obtained in a common Cartesian coordinate system is presented and examined systematically. We also derive the reason why stress tensor variability can be adequately represented by variability of distinct tensor components in a multivariate manner.

In the present paper, the multivariate distribution model of complete tensor components is presented first, and difficulties faced in application to the analysis of stress variability discussed. To overcome these difficulties the multivariate distribution of distinct tensor components is introduced. We then analytically demonstrate the transformational consistency and invariance of this statistical distribution model in terms of mean, covariance matrix and probability density function (PDF). Finally, using actual *in situ* stress data, the multivariate characteristics of stress data is confirmed and inappropriateness of a quasi-tensorial distribution model discussed. Some relevant contents and derivations are shown in the Appendices. The notation adopted here generally follows the convention of bold uppercase, bold lowercase and normal lowercase letters denoting matrix, vector and scalar, respectively, unless otherwise noted.

2. Multivariate distribution model

Generally, a tensor is a quantity that can be represented by an organised array of numerical values. The order of a tensor is the dimension of the array needed to represent it, or equivalently the number of indices needed to label a component of that array. Thus, scalars, being single numbers, are zero order tensors, and vectors, being one-dimensional arrays, are tensors of the first order. Stress tensors are represented by 2×2 or 3×3 two-dimensional arrays, and therefore are second order. Unless otherwise noted, here the term "tensor" is specifically used to denote a symmetric 2×2 or 3×3 second order tensor. As stress is a second order tensor, the explicit and intuitive approach to characterise stress variability is to use a matrix variate distribution, as these characterise the variability of matrices by considering each matrix as a single entity.[51] However, current limitations of and application difficulties associated with matrix variate statistics require the more applicable approach of multivariate statistics to be used, as the two techniques can be shown to be equivalent.[51]

Here, we first introduce the matrix variate normal distribution to demonstrate the equivalence between the matrix variate statistics of a stress tensor and the multivariate statistics of the complete tensor components. Then, by making use the symmetric structure of the stress tensor and to avoid the singularity caused by repeated rows and