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Fourier spectral methods for numerical solving the problem of boundary control of the linear wave equation

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Abstract In this paper, a Fourier spectral method is used to reduce the optimal boundary control problem for a two-dimensional wave equation to a countable number of control problems for a one-dimensional wave equation which are transformed to the optimal control problems with integral constraints using the Laplace transform. The numerical integration and differentiation methods are used to approximate the resulting problems with quadratic programming problems. An illustrative numerical example is presented to indicate the efficiency of the proposed method.

1. Introduction

Let Ω be a bounded domain in R^m and $T = (0, \tau)$. Consider a wave equation

$$u_{tt}(z,t) = \alpha^2 \Delta u(z,t), \quad (z,t) \in \Omega \times T, \tag{1}$$

with the initial conditions

$$u(z,0) = f(z), \quad u_t(z,0) = h(z), \quad z \in \Omega, \tag{2}$$

and the boundary conditions

$$u(z,t) = l(z,t), \quad (z,t) \in \Gamma \times T, \tag{3}$$

where Γ is a nonempty open set of $\partial\Omega$. If we impose an extra boundary condition $u(z,t) = 0, (z,t) \in (\partial\Omega - \Gamma) \times T$, then the existence and uniqueness of a solution for (1)–(3) is guaranteed under certain assumptions on domain Ω and functions $f(z), h(z)$ and $l(z,t)$ [1]. For the description of the desired target state, we add later on the following end conditions

$$u(z,\tau) = g(z), \quad u_t(z,\tau) = k(z), \quad z \in \Omega. \tag{4}$$

Here we assume that our control problem consists of finding the control $l(z,t)$, which minimizes the following functional

$$J(l) = \int_0^\tau \int_\Gamma u^2(z,t)\,dz\,dt. \tag{5}$$

The optimal control problems for wave equations have wide applications in robotics, control of space constructions, etc. This type of problems has been studied by many authors. For example in [2] a multi-objective optimal control problem for one-dimensional wave equations is transformed into a multi-objective linear programming problem and a Pareto optimal solution is derived using the simulated annealing (SA) metaheuristic. The problem of minimizing a quadratic functional on trajectories of the wave equation using the

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Fourier method is considered in [3] where the density of external forces is considered as a control function. Optimal control of the linear wave equation using the measure theoretical approach has been studied in [4].

The spectral method has been used in computing control problems [5,6]. Moreover, in recent years, this method has been used to solve distributed optimal control problems [3,7,8]. The spectral method enjoys a great superiority of fast convergence rate when the solutions are smooth, which is vital to successful approximation of optimal control problems. In general, the solutions to the optimal control problems have limited regularity due to, e.g., the constraints, and therefore spectral accuracy generally cannot be achieved. Consequently, the spectral method is not so widely used in solving constrained distributed optimal control problems [8]. According to our knowledge, most problems solved by the spectral methods have homogenous boundary conditions, which would be a restrictive assumption. While in this paper we use spectral method to solve the Problem (1)–(5) with the boundary condition as the control function, which may be different from other works in this sense.

The rest of the paper is organized as follows. In next Section Problem (1)–(5) is reduced into a countable number of problems for a one-dimensional problem. In Section 3, an explicit solution is proposed to the resulting problems and a sequence of optimal control problems is achieved using the Laplace transform. Conversion of the optimal control problems to the quadratic programming problem is the subject of Section 4. A numerical example is carried out in Section 5. The last section is the conclusion.

2. The spectral method for two-dimensional problem

The fundamental idea behind spectral methods is to approximate solutions of PDEs by finite series of orthogonal functions such as the complex exponentials, Chebyshev or Legendre polynomials. For an easy understanding, we consider a two-dimensional problem on rectangular domain $\Omega = (0,a)\times(0,b)$ with $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = [0,a]\times\{0\}$ and $\Gamma_2 = [0,a]\times\{b\}$. Let

$$u|_{y=0} = w(x,t), \quad u|_{y=b} = v(x,t). \tag{6}$$

An appropriate basis to approximate the solution of (1)–(4) is the Fourier cosine Basis $\{\cos(\lambda_n x)\}$ where, $\lambda_n = \frac{n\pi}{a}$. We expand the solution $u(x,y,t)$ in the series with respect to this basis:

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(y,t)\cos(\lambda_n x). \tag{7}$$

Note that if we impose an extra boundary condition $u(z,t) = 0, (z,t) \in (\partial\Omega - \Gamma)\times T$, then the sinus basis is appropriate. Because with this choice for basis, this additional condition is satisfied, automatically.

Denote the sum of the first $N+1$ elements of (7) by $u_N(x,y,t)$. We now show the decay rate of the error increases with the number N. If $u(x,y,t)$ is such that we can substitute it by its expansion series (7), the error in approximating u by u_N can be measured by the size of the tail of the above series, given by $\varepsilon_N = ||u-u_N|| = ||\sum_{k=N+1}^{\infty} u_k(y,t)\cos(\lambda_k x)||$. If $u(x,y,t)$ is infinitely differentiable with respect to x, then by using integration by parts we have, $u_k(y,t) = (-1)^m \frac{2a^{m-1}}{k^m\pi^m}\int_0^a \frac{\partial^m u}{\partial x^m}(x,y,t)\cos(\lambda_k x)dx, m = 1,2,\ldots$. This means that the decay of the spectral coefficients is faster than any negative power of m, or that $\varepsilon_N = O(e^{-cm})$, for some $c>0$.

Assume that $f,h,g,k,w,v \in L_2$. Expand these functions in series with respect to the cosine basis:

$$f(x,y) = \sum_{n=0}^{\infty} f_n(y)\cos(\lambda_n x),$$
$$w(x,t) = \sum_{n=0}^{\infty} w_n(t)\cos(\lambda_n x),\ldots. \tag{8}$$

Inserting (7) and (8) in (1)–(4) and using the orthogonality of the cosine basis, the two-dimensional wave Eqs. (1)–(4) reduces to a one-dimensional wave equation

$$\frac{\partial^2 u_n(y,t)}{\partial t^2} = \frac{\partial^2 u_n(y,t)}{\partial y^2} - \lambda_n^2 u_n(y,t), \tag{9}$$

with the initial conditions,

$$u_n(y,0) = f_n(y), \; \frac{\partial u_n(y,0)}{\partial t} = h_n(y), \tag{10}$$

and the boundary conditions,

$$u_n(0,t) = w_n(t), \; u_n(b,t) = v_n(t), \tag{11}$$

and the end conditions,

$$u_n(y,\tau) = g_n(y), \; \frac{\partial u_n(y,\tau)}{\partial t} = k_n(y). \tag{12}$$

Obviously the Fourier coefficients of w and v, i.e., $w_n(t)$ and $v_n(t)$ are control functions depending on time only. It is easy to see that

$$w_n(0) = f_n(0), \; w_n(\tau) = g_n(0), \; v_n(0) = f_n(b), \; v_n(\tau) = g_n(b),$$

$$\dot{v}_n(0) = h_n(b), \; \dot{v}_n(\tau) = k_n(b), \; \dot{w}_n(0) = h_n(0),$$
$$\dot{w}_n(\tau) = k_n(0). \tag{13}$$

Moreover, the cost functional (5) takes the following form

$$J_n(w_n, v_n) = \int_0^{\tau}\{w_n^2(t) + v_n^2(t)\}dt. \tag{14}$$

Obviously, we have reduced the dimension of the problem into a countable number of one-dimensional problems which are typically easier to handle numerically. Laplace transforms are useful in solving optimal control problems in linear distributed parameter systems [2,9]. An application of this tool to solve Problem (9)–(14) is presented in next section.

3. Laplace transform and control problem

We now proceed to solve the PDE (9)–(12) applying the Laplace transform with respect to t which yields,

$$\bar{u}_n''(y,s) - \{s^2 + \lambda_n^2\}\bar{u}_n(y,s) = -sh_n(y) - f_n(y), \tag{15}$$

$$\bar{u}_n(0,s) = \bar{w}_n(s), \; \bar{u}_n(b,s) = \bar{v}_n(s), \tag{16}$$

where the Laplace transform of a function is denoted by a bar placed over the symbol for the function. A solution to differential Eq. (15) with initial condition (16) is

$$\bar{u}_n(y,s) = \bar{w}_n(s)\bar{z}_n(b-y,s) + \bar{v}_n(s)\bar{z}_n(y,s) + \bar{\mu}_n(y,s), \tag{17}$$

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