ARTICLE IN PRESS

Alexandria Engineering Journal (2017) xxx, xxx–xxx

HOSTED BY

ELSEVIER

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

ORIGINAL ARTICLE

Analysis of solution for system of nonlinear fractional Burger differential equations based on multiple fractional power series

Bahram Agheli [a,*], Rahmat Darzi [b]

[a] Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
[b] Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran

Received 9 October 2016; revised 21 December 2016; accepted 27 December 2016

KEYWORDS

System of fractional Burger equations;
New homotopic perturbation method;
Caputo derivative

Abstract We have applied the new approach of homotopic perturbation method (*NHPM*) for Burger differential system equations featuring time-fractional derivative. A combination of *NHPM* and multiple fractional power series form has been used the first time to present analytical solution. In order to illustrate the simplicity and ability of the suggested approach, some specific and clear examples have been given. All numerical calculations in this manuscript have been carried out with *Mathematica*.

1. Introduction

In this research work, it has been proposed that the new *HPM* based on the multiple fractional power series can be engaged to answer of Burger differential system equations featuring time-fractional derivative.

This system equation has frequently appeared in different fields of science and engineering such as physics, optics, plasma physics, superconductivity and quantum mechanics [1].

There are some more books related to fractional calculus for interested readers [2,3]. It should be noted that there are no accurate analytical solutions for most fractional differential equations. Consequently, for such equations we have to employ some direct and iterative methods. Researchers have used various methods to solve systems equations in recent years. Some familiar methods are as follows: variational iteration method [4–6], homotopic perturbation method [7,8], homotopic analysis method [9,10] and so on [11–21].

This paper is organized as follows: in Section 2, fundamental idea of the new method is presented. Convergence of this method is explained in Section 3. In Section 4, the application of innovative *HPM* to Burger differential system equations featuring time-fractional derivative is illustrated, and some numerical examples are presented. And conclusions are drawn in Section 5.

2. Fundamental idea of the new method

To describe the fundamental ideas of the *NHPM* method for Burger differential system equations featuring time-fractional derivative:

* Corresponding author.
E-mail address: b.agheli@qaemiau.ac.ir (B. Agheli).

Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.12.021
1110-0168

Please cite this article in press as: B. Agheli, R. Darzi, Analysis of solution for system of nonlinear fractional Burger differential equations based on multiple fractional power series, Alexandria Eng. J. (2017), http://dx.doi.org/10.1016/j.aej.2016.12.021

$$\begin{cases} D_\tau^\mu u - u_{\zeta\zeta} + \eta u u_{\zeta\zeta} + \vartheta(uv)_s = h(\zeta,\tau), & 0 < \mu \leqslant 1 \\ D_\tau^\mu v - v_{\zeta\zeta} + \lambda v v_{\zeta\zeta} + \mu(uv)_s = g(\zeta,\tau), \end{cases} \tag{2.1}$$

with the following initial condition:

$$u(\zeta,\tau_0) = \varphi(\zeta), \quad v(\zeta,\tau_0) = \psi(\zeta), \tag{2.2}$$

where η, ϑ, λ and μ are constants, D^μ denotes that Caputo fractional and $h(\zeta,\tau)$ and $g(\zeta,\tau)$ are inhomogeneous terms.

For obtain the solution of (2.1), by using *NHPM*, we make the under homotopic:

$$\begin{cases} (1-q)\left(D_\tau^\mu U - u_0\right) + q\left(D_\tau^\mu U - u_{\zeta\zeta} + \eta u u_{\zeta\zeta} + \vartheta(uv)_s - h(\zeta,\tau)\right) = 0 \\ (1-q)\left(D_\tau^\mu V - v_0\right) + q\left(D_\tau^\mu V - v_{\zeta\zeta} + \lambda v v_{\zeta\zeta} + \mu(uv)_s - g(\zeta,\tau)\right) = 0. \end{cases} \tag{2.3}$$

or

$$\begin{cases} D_\tau^\mu U = u_0 - q\left(u_0 - u_{\zeta\zeta} + \eta u u_{\zeta\zeta} + \vartheta(uv)_s - h(\zeta,\tau)\right) \\ D_\tau^\mu V = v_0 - q\left(v_0 - v_{\zeta\zeta} + \lambda v v_{\zeta\zeta} + \mu(uv)_s - g(\zeta,\tau)\right). \end{cases} \tag{2.4}$$

Using the inverse operator, $L^{-1} = I_\tau^\mu(.)$ to both sides of (2.4), then we gain

$$\begin{cases} U(\zeta,\tau) = U(\zeta,\tau_0) + I_\tau^\mu u_0 - q I_\tau^\mu\left(u_0 - u_{\zeta\zeta} + \eta u u_{\zeta\zeta} + \vartheta(uv)_s - h(\zeta,\tau)\right) \\ V(\zeta,\tau) = V(\zeta,\tau_0) + I_\tau^\mu v_0 - q I_\tau^\mu\left(v_0 - v_{\zeta\zeta} + \lambda v v_{\zeta\zeta} + \mu(uv)_s - g(\zeta,\tau)\right). \end{cases} \tag{2.5}$$

where $U(\zeta,\tau_0) = u(\zeta,\tau_0)$ and $V(\zeta,\tau_0) = v(\zeta,\tau_0)$.

Now assume we introduce the solution of (2.5) in the next form

$$\begin{cases} U(\zeta,\tau) = U_0(\zeta,\tau) + q\,U_1(\zeta,\tau) + q^2\,U_2(\zeta,\tau) + \cdots \\ V(\zeta,\tau) = V_0(\zeta,\tau) + q\,V_1(\zeta,\tau) + q^2\,V_2(\zeta,\tau) + \cdots, \end{cases} \tag{2.6}$$

where $U_k(\zeta,\tau),\ V_k(\zeta,\tau),\ k = 0,1,2,3,\ldots$, are functions which should be calculated.

Definition 2.1. A series expansion of the formula below

$$\sum_{m=0}^{\infty} c_m(\tau-\tau_0)^{m\mu} = c_0 + c_1(\tau-\tau_0)^{\mu} + c_2(\tau-\tau_0)^{2\mu} + \cdots,$$
$$0 \leqslant n-1 < \mu \leqslant n,\ \tau_0 \leqslant \tau,$$

is called fractional power series around $\tau = \tau_0$.

Definition 2.2. A series expansion of the next form

$$\sum_{m=0}^{\infty} f_m(\zeta)(\tau-\tau_0)^{m\mu}, \quad 0 \leqslant n-1 < \mu \leqslant n,\ \tau_0 \leqslant \tau,$$

is called multiple fractional power series around $\tau = \tau_0$.

The equations given below show the primary approximation of the answer of (2.1)

$$u_0(\zeta,\tau) = \sum_{k=0}^{\infty} a_k(\zeta)\,q_k(\tau), \quad v_0(\zeta,\tau) = \sum_{k=0}^{\infty} b_k(\zeta)\,q_k(\tau), \tag{2.7}$$

where $a_k(\zeta),\ b_k(\zeta),\ k = 0,1,2,3,\ldots,$ are unfamiliar coefficients, and $q_k(\tau),\ k = 0,1,2,3,\ldots$, are particular functions.

It is deserving to consider that if $h(\zeta,\tau)$, and $u_0(\zeta,\tau)$ are analytic around $\tau = 0$, then their Taylor series can be written as

$$u_0(\zeta,\tau) = \sum_{k=0}^{\infty} a_k(\zeta)\,\tau^{k\mu}, \quad v_0(\zeta,\tau) = \sum_{k=0}^{\infty} b_k(\zeta)\,\tau^{k\mu}. \tag{2.8}$$

With considering (2.5) and substituting (2.6) and (2.7) into that and equating the coefficients of the same power q

$$\begin{aligned}
q^0 &: \begin{cases} U_0(\zeta,\tau) = \varphi(\zeta) + \sum_{k=0}^{\infty} a_k(\zeta) I_\tau^\mu(q_k(\tau)) \\ V_0(\zeta,\tau) = \psi(\zeta) + \sum_{k=0}^{\infty} b_k(\zeta) I_\tau^\mu(q_k(\tau)) \end{cases} \\
q^1 &: \begin{cases} U_1(\zeta,\tau) = -\sum_{k=0}^{\infty} a_k(\zeta) I_\tau^\mu(q_k(\tau)) - I_\tau^\mu(N(U_0(\zeta,\tau) - h(\zeta,\tau))) \\ V_1(\zeta,\tau) = -\sum_{k=0}^{\infty} b_k(\zeta) I_\tau^\mu(q_k(\tau)) - I_\tau^\mu(N(V_0(\zeta,\tau) - g(\zeta,\tau))) \end{cases} \\
q^2 &: \begin{cases} U_2(\zeta,\tau) = -I_\tau^\mu(N(U_0(\zeta,\tau), U_1(\zeta,\tau))) \\ V_2(\zeta,\tau) = -I_\tau^\mu(N(V_0(\zeta,\tau), V_1(\zeta,\tau))) \end{cases} \\
&\ldots \\
q^k &: \begin{cases} U_k(\zeta,\tau) = -I_\tau^\mu(N(U_0(\zeta,\tau), U_1(\zeta,\tau)\ldots, U_{k-1}(\zeta,\tau))) \\ V_k(\zeta,\tau) = -I_\tau^\mu(N(V_0(\zeta,\tau), V_1(\zeta,\tau)\ldots, V_{k-1}(\zeta,\tau))). \end{cases}
\end{aligned} \tag{2.9}$$

By solving these equations in such a way that $U_1(\zeta,\tau) = 0$ and $V_1(\zeta,\tau) = 0$, then Eq. (2.9) yield to $U_k(\zeta,\tau) = 0,\ V_k(\zeta,\tau) = 0,\ k = 2,3,\ldots$.

As a result, the numerical analytical solution may be gained:

$$\begin{cases} u(\zeta,\tau) = U_0(\zeta,\tau) = g(\zeta) + \sum_{k=0}^{\infty} a_k(\zeta)\left(I_\tau^\mu q_k(\tau)\right) \\ u(\zeta,\tau) = U_0(\zeta,\tau) = g(\zeta) + \sum_{k=0}^{\infty} a_k(\zeta)\left(I_\tau^\mu q_k(\tau)\right). \end{cases} \tag{2.10}$$

It should be noted that if $h(\zeta,\tau),\ g(\zeta,\tau),\ u_0(\zeta,\tau)$ and $v_0(\zeta,\tau)$ are analytic around $\tau = \tau_0$, then Taylor series can be written as

$$u_0(\zeta,\tau) = \sum_{k=0}^{\infty} a_k(\zeta)(\tau-\tau_0)^{k\mu}, \quad v_0(\zeta,\tau) = \sum_{k=0}^{\infty} b_k(\zeta)(\tau-\tau_0)^{k\mu},$$
$$h(\zeta,\tau) = \sum_{k=0}^{\infty} a_k^\star(\zeta)(\tau-\tau_0)^{k\mu}, \quad g(\zeta,\tau) = \sum_{k=0}^{\infty} b_k^\star(\zeta)(\tau-\tau_0)^{k\mu},$$

can be used in Eq. (2.9), where $a_k(\zeta),\ b_k(\zeta),\ k = 0,1,2,\ldots$, are unknown coefficients which must be computed, and $a_k^\star(\zeta),\ b_k^\star(\zeta),\ k = 0,1,2,\ldots$, are known ones.

3. Convergence analysis

A large number of problems can be treated by *NHPM* through utilizing the methodology that has been elaborated in the previous sections [22].

Theorem 3.1. *Presume that S and $T$are Banach spaces and $\mathcal{A} : S \to T$ is a contractive nonlinear mapping which is*

$$\forall\, \upsilon, \upsilon^\star \in S; \quad \left\|\mathcal{A}(\upsilon) - \mathcal{A}(\upsilon^\star)\right\| \leqslant \lambda\left\|\upsilon - \upsilon^\star\right\|,\ 0 < \lambda < 1.$$

Then, due to Banach's fixed point theorem $\mathcal{A}$, has a unique fixed point u, which is $\mathcal{A}(u) = u$. Assume that the sequence provided by new HPM is stated that

Please cite this article in press as: B. Agheli, R. Darzi, Analysis of solution for system of nonlinear fractional Burger differential equations based on multiple fractional power series, Alexandria Eng. J. (2017), http://dx.doi.org/10.1016/j.aej.2016.12.021