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# Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity

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ABSTRACT

In our previous two works, we studied the blow-up and lifespan estimates for damped wave equations with a power nonlinearity of the solution or its derivative, with scattering damping independently. In this work, we are devoted to establishing a similar result for a combined nonlinearity. Comparing to the result of wave equation without damping, one can say that the scattering damping has no influence.



## 1. Introduction

Recently, the small data Cauchy problem of damped semilinear wave equations with time dependent variable coefficients attracts more and more attention. The works of Wirth [1–3] showed that the behavior of the solution of the following linear problem

$$\begin{cases} u^0_{tt} - \Delta u^0 + \dfrac{\mu}{(1+t)^\beta} u^0_t = 0, \quad \text{in } \mathbf{R}^n \times [0,\infty), \\ u^0(x,0) = u_1(x),\ u^0_t(x,0) = u_2(x), \quad x \in \mathbf{R}^n, \end{cases}$$

heavily relies on the decay rate $\beta$ and the size of the positive constant $\mu$. Then people get interested in the corresponding nonlinear problem, i.e. the following small data Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u|^p \quad \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), \quad x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

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where $\mu > 0$, $n \in \mathbf{N}$ and $\beta \in \mathbf{R}$, and $\varepsilon$ measures the smallness of the data. Before going on, it is necessary to mention two corresponding nonlinear problems without damping

$$\begin{cases} u_t - \Delta u = |u|^p & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u_{tt} - \Delta u = |u|^p & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.3}$$

For Cauchy problem (1.2) we know that it admits the critical value of $p$ by

$$p_F(n) := 1 + \frac{2}{n},$$

which is so-called Fujita exponent, while the one for problem (1.3) is so-called Strauss exponent $p_S(n)$, which is the positive root of the quadratic equation,

$$\gamma(p,n) := 2 + (n+1)p - (n-1)p^2 = 0.$$

**Remark 1.1.** "critical" here means the borderline which divides the domain of $p$ into the blow-up part and the global existence part of the solution.

**Remark 1.2.** It is easy to prove that

$$p_F(n) < p_S(n) \quad \text{for } n \geq 2.$$

Now we come back to Cauchy problem (1.1). It is interesting to study the relation of the critical exponents among (1.1)–(1.3). For $\beta \in [-1,1)$, due to the works [4–9], we know that it admits the same critical exponent as that of problem (1.2). For $\beta > 1$, since the authors showed blow-up result for $1 < p < p_S(n)$ in [10], we may believe that it has the same critical exponent as that of (1.3).

If we consider the case $\beta = 1$ for Cauchy problem (1.1), the size of the positive constant $\mu$ should also be taken into account. Generally speaking, if $\mu$ is relatively large, the term $\{\mu/(1+t)\}u_t$ in the equation will have the main influence on the behavior of the solution, which means that this case has the same critical exponent as that of problem (1.2). See the works [11,12]. But, if $\mu$ is relatively small, we may conjecture that the influence of $u_{tt}$ will dominate over $\{\mu/(1+t)\}u_t$, which means that the critical exponent is related to $p_S(n)$. See the work [13] by the authors and Wakasa for $0 < \mu < (n^2+n+2)/\{2(n+2)\}$, which was extended to $0 < \mu < (n^2+n+2)/(n+2)$ by Ikeda and Sobajima [14] and Tu and Lin [15,16]. Unfortunately, till now we are not clear of the boardline of $\mu$, which determines that the critical power of Cauchy problem (1.1) with $\beta = 1$ will be Fujita or Strauss. We refer the reader to a very recent work by Palmieri and Reissig [17].

In a recent work [18] by the authors, we study the blow-up for the small data Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u_t|^p & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.4}$$

If $\beta > 1$, then we showed that the problem has no global solution for $1 < p \leq p_G(n)$, where

$$p_G(n) := \frac{n+1}{n-1},$$

which denotes the critical exponent for Glassey conjecture. In this work, we are devoted to studying the small data Cauchy problem with combined nonlinear terms, that is:

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u_t|^p + |u|^q & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.5}$$