Nonlinear Analysis: Real World Applications 45 (2019) 226–238

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Exact controllability of the wave equation with time-dependent and variable coefficients

Yu-Xiang Liu*

School of Science, Qingdao University of Technology, Qingdao, Shandong 266520, PR China
Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

A R T I C L E I N F O

Article history:
Received 8 April 2014
Received in revised form 4 May 2017
Accepted 6 July 2018

Keywords:
Wave equation with variable coefficients
Exact controllability
Riemannian manifold

A B S T R A C T

This paper deals with boundary exact controllability for the dynamics governed by the wave equation with variable coefficients in time and space, subject to Dirichlet or Neumann boundary controls. The observability inequalities are established by the Riemannian geometry method under some geometric conditions.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega = \Gamma$. It is assumed that Γ consists of two parts: Γ_0 and Γ_1, $\Gamma_0 \cup \Gamma_1 = \Gamma$, with Γ_0 nonempty and relatively open in Γ. Let Q be the finite cylinder $\Omega \times [0, T]$ with lateral boundary $\Sigma = \Gamma \times [0, T]$. We consider the exact controllability for the mixed problem

$$\begin{cases} y_{tt} + \mathcal{A}(t, x)y = 0 & \text{in} \quad Q, \\ y(0) = y^0, \quad y_t(0) = y^1 & \text{on} \quad \Omega, \end{cases} \tag{1.1}$$

with Neumann boundary control

$$y_{\nu_{\mathcal{A}}} = \upsilon \quad \text{on} \quad \Gamma \times [0, T], \tag{1.1a}$$

or Dirichlet boundary control

$$y = \upsilon, \quad \text{on } \Gamma_0 \times [0, T], \qquad y = 0, \quad \text{on } \Gamma_1 \times [0, T], \tag{1.1b}$$

* Correspondence to: School of Science, Qingdao University of Technology, Qingdao, Shandong 266520, PR China.
E-mail address: yxliu@amss.ac.cn.

https://doi.org/10.1016/j.nonrwa.2018.07.005
1468-1218/

where y_{tt} stands for $\partial^2 y/\partial t^2$, $a_{ij} = a_{ji}$ are C^∞ functions in R^n, $\nu(x)$ is the unit exterior normal vector at $x \in \Gamma$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2, \qquad x \in \Omega, \tag{1.2}$$

$$\mathcal{A}(t,x) = -\beta(t)\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i}), \tag{1.3}$$

for some constant $a > 0$.

We ask whether there is some constant $T_0 > 0$ such that if $T > T_0$, the following steering property of (1.1) and (1.1a), or (1.1) and (1.1b) holds true: for all initial data $y^0, y^1 \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a suitable control function υ, whose corresponding solution of (1.1) and (1.1a), or (1.1) and (1.1b) satisfies

$$y(\cdot,T) \equiv y_t(\cdot,T) \equiv 0. \tag{1.4}$$

When the answer is in the affirmative, we then say that the dynamics (1.1) and (1.1a), or (1.1) and (1.1b) is exactly controllable in the interval $[0,T]$ on $L^2(\Omega) \times H^{-1}(\Omega)$ by means of the Neumann boundary condition or Dirichlet control function , respectively.

This problem has received considerable attention in the literature, with numerous contributions achieved over the past several years. For the constant coefficient case($\beta(t) = 1$, $a_{ij}(x) = \delta_{ij}$), we refer to [1–6], and references there.

In the present paper, we consider the observability inequality for system (1.1) and (1.1a), or (1.1) and (1.1b) by the Riemannian geometry method. This method was introduced by [7] to deal with controllability for variable coefficients wave equation and extended in [8–16] and references there. For a survey on the differential geometric methods, see [17] and [18]. Several multiplier identities, which have been built for the wave equation with constant coefficients, are generalized to the variable coefficient case by some computational techniques in Riemannian geometry and then observability inequalities are derived from those identities.

Here we shall combine [7] and [19] to obtain observability inequalities for system (1.1) and (1.1a), or (1.1) and (1.1b) to establish the corresponding exact controllability.

Our paper is divided on three sections. In Section 2, we give notation and state the principle results. In Section 3, we give multiplier identities and prove the principle results.

2. Notations and main results

Suppose that

$$\sum_{ij=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \qquad \forall x \in R^n, \quad \xi = (\xi_1,\xi_2,\ldots,\xi_n)^\tau \in R^n, \quad \xi \neq 0. \tag{2.1}$$

Set

$$A(x) = (a_{ij}(x)). \tag{2.2}$$

We introduce

$$g(x) = (g_{ij}(x)) = A^{-1}(x) \quad \text{for} \quad x \in R^n \tag{2.3}$$