Nonlinear Analysis: Real World Applications 45 (2019) 255–261

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

A remark on the global existence of weak solutions to the compressible quantum Navier–Stokes equations

Tong Tang[a,*], Zujin Zhang[b]

[a] *Department of Mathematics, College of Sciences, Hohai University, Nanjing 210098, PR China*
[b] *College of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, PR China*

ARTICLE INFO

Article history:
Received 10 September 2017
Received in revised form 4 July 2018
Accepted 9 July 2018

Keywords:
Quantum Navier–Stokes equations
Global existence
Weak solutions

ABSTRACT

In this paper, we obtain the global existence of weak solutions to the compressible quantum Navier–Stokes equations. By virtue of a useful identity and an interesting estimate, we solve the critical case that the viscosity equals the dispersive coefficient. This result removes the restrictions on the coefficients and improves the recent work of Antonell and Spirito (2017) in some senses.

1. Introduction

In this note, we are interested in the following compressible quantum Navier–Stokes equations, also called the Navier–Stokes–Korteweg (N–S–K) equations, in $\Omega = \mathbb{T}^N (N = 2, 3)$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = 2\nu \operatorname{div}(\rho D(u)) + 2\kappa^2 \rho \nabla (\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}), \end{cases} \tag{1.1}$$

with initial data

$$\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0, \tag{1.2}$$

where $\rho = \rho(t, x)$ and $u = u(t, x)$ respectively denote the density and velocity. ρ and u are periodic in Ω, $D(u) = \frac{1}{2}[\nabla u + \nabla^t u]$ is the strain tensor. The physical parameters are the viscosity constant ν and the capillary constant κ.

Quantum mechanics is one of the most significant subject on the contemporary scientific research. In the literatures, there are a wide range of studies in the quantum model, such as [1–8]. Recently, a large

* Corresponding author.
E-mail addresses: tt0507010156@126.com (T. Tang), zhangzujin361@163.com (Z. Zhang).

https://doi.org/10.1016/j.nonrwa.2018.07.009
1468-1218/

number of mathematicians study the compressible N–S–K system due to its physical importance, complexity, rich phenomena and mathematical challenges. The N–S–K model describes the variation of density at the interfaces between two phases, generally a liquid–vapor mixture. Readers can refer to [9–14] to find more physical illustrations.

It is well-known that the existence of the weak solutions to the compressible Navier–Stokes equations is an open problem, especially when viscosity coefficients depend on density. The existence is generally obtained by a constructive method: constructing approximate solutions (e.g. by Galerkin method or finite differences), then passing to limit by the analysis of compactness. We suggest readers can refer to [15–18] for more details. However, regarding to N–S–K system, the construction of approximation solution is a rough and arduous issue. The method used in compressible Navier–Stokes equations cannot be applied to N–S–K system directly, owing to the strongly nonlinear third-order dispersive structure in the momentum equation. More precisely, it is hard to obtain the Mellet–Vasseur type estimate in the N–S–K system. This type estimate was first discovered by Mellet and Vasseur [17] and it is extremely vital in the process of compactness. Recently, Antonell and Spirito [19] used some technical skills to construct the approximating system and proved the global existence of weak solutions with $\kappa < \nu$. Their main idea is constructing approximating system with cold pressure and damping term, then using an effective velocity w to change the approximating system into a new formulation without the dispersive term and cold pressure. It is a remarkable fact that the method depends on the relation $\kappa < \nu$, otherwise we cannot gain the desired B–D entropy estimate and higher integrability of $\rho u^{2+\delta}$.

Motivated by the work [3], we rewrite the dispersive term and use an interesting identity to overcome the obstacle. Therefore, the restriction on κ and ν can be removed, that means the same result can be obtained under the critical case $\kappa = \nu$. The definition of weak solution to (1.1) and assumption on initial data is the same as [19], and we state our main result in the following:

Theorem 1.1. *Let $\nu = \kappa > 0$. In the two dimensional case, for any $\gamma > 1$ and $0 < T < \infty$, there exist global weak solutions to system (1.1) on $(0, T) \times \mathbb{T}^2$; In the three dimensional case, for any $1 < \gamma < 3$ and $0 < T < \infty$, there exist global weak solutions to system (1.1) on $(0, T) \times \mathbb{T}^3$.*

We will follow the framework of [19], and only sketch the main ideas for the proofs. Readers can refer to [19] for more details.

2. Approximating system

In this section, we construct the approximating system as [19], and use an interesting identity to change the dispersive term into a new formulation. The approximating system is:

$$\begin{cases} \partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon u_\epsilon) = 0, \\ \partial_t(\rho_\epsilon u_\epsilon) + \operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) + \nabla\big(\rho_\epsilon^\gamma + p(\rho_\epsilon)\big) + \widetilde{p}(\rho_\epsilon) u_\epsilon = 2\nu \operatorname{div}\mathbb{S}_\epsilon + \kappa^2 \operatorname{div}\mathbb{K}_\epsilon, \end{cases} \tag{2.1}$$

where

$$\operatorname{div}\mathbb{S}_\epsilon = \operatorname{div}\big(h(\rho_\epsilon) D(u_\epsilon)\big) + 2\nu \nabla\big(g(\rho_\epsilon)\operatorname{div}u_\epsilon\big),$$
$$\operatorname{div}\mathbb{K}_\epsilon = 2\rho_\epsilon \nabla\Big(\frac{h'(\rho_\epsilon)\operatorname{div}(h'(\rho_\epsilon)\nabla\sqrt{\rho_\epsilon})}{\sqrt{\rho_\epsilon}}\Big).$$

The coefficient $\widetilde{p}(\rho_\epsilon)$ is defined as

$$\widetilde{p}(\rho_\epsilon) = e^{-\frac{1}{\epsilon^4}}(\rho_\epsilon^{-\frac{1}{\epsilon^2}} + \rho_\epsilon^{\frac{1}{\epsilon^2}}),$$