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On the dynamics of the Rayleigh–Duffing oscillator

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ABSTRACT

We give a complete algebraic characterization of the first integrals of the Rayleigh–Duffing oscillator. We prove the non existence of centers of such system and we study the form of the singular first integrals at the origin.

1. Introduction and statement of the main results

The Rayleigh–Duffing oscillator can be formulated as the second-order differential equation

$$\ddot{x} + ax + x^3 + 2b\dot{x} + (\dot{x})^3 = 0, \tag{1}$$

where a, b are parameters. We recall that the dot is the derivative with respect to the independent variable t, that x is the displacement, a is the stiffness and b is related with the linear damping. This equation was introduced by Lord Rayleigh in his famous paper in acoustics, see [1]. Rayleigh treated the self-sustained vibration of a clarinet reed. He was aware that the vibration of the reed is affected by a damping force which he described in the form of a cubic function of $\dot{x}$. This second-order differential equation is equivalent to the first order differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax - 2by - x^3 - y^3. \end{aligned} \tag{2}$$

Note that system (2) is symmetric with respect to the transformation $(x, y) \mapsto (-x, -y)$. The global dynamics of Rayleigh–Duffing oscillators with global parameters, including singular points, existence and coexistence

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of limit cycles and homoclinic loops was studied in [2,3]. The forced Rayleigh–Duffing equation is an example of a dynamical system that exhibits chaotic behavior, see for instance [4].

Chaos is essentially a dynamical concept, in which two initially close trajectories may be divergent after a finite time. Contrary to intuition, chaotic behavior is not an exclusive property of dissipative or random systems but of conservative and deterministic systems as well. Focusing on the last ones, for non-integrable systems, the irregular behavior that a system may exhibit is associated with the appearance of chaos, because it seems impossible to predict the long-term evolution of a non-integrable system—at least with the most commonly used perturbation techniques. To the contrary, if the system is integrable then the existence of constants of motion (also called integrals and hence the name) are responsible for the regular evolution of the phase-space trajectories of the system in well-defined regions of the phase space. More concretely, for a planar differential system the existence of a first integral reduces the complexity of its dynamics because it solves completely the problem (at least theoretically) of determining its phase portrait. In general for a given differential system it is a difficult problem to determine the existence or non-existence of first integrals. We recall that a function $H(x, y)$ is a *first integral* of a planar differential if it is continuous and defined on a full Lebesgue measure subset $\Omega \subseteq \mathbb{R}^2$, is not locally constant on any positive Lebesgue measure subset of Ω and moreover is constant along each orbit in Ω of that system.

During recent years the interest in the study of the integrability of differential equations has attracted much attention from the mathematical community. Darboux theory of integrability plays a central role in the integrability of the polynomial differential systems since it gives a sufficient condition for the integrability inside a wide family of functions. We highlight that it works for real or complex polynomial differential systems and that the study of complex algebraic solutions is necessary for obtaining all real first integrals of a real polynomial differential system. The first main result in the paper is the following.

Theorem 1. *The following holds for system (2):*

(a) *It has no global C^∞ first integrals and so no global analytic first integrals.*
(b) *It has no Darboux polynomials.*
(c) *It is not Darboux integrable.*
(d) *It is not Liouville integrable.*

We recall that a Darboux first integral is a product of complex powers of Darboux polynomials and exponential factors (for a precise definition, see the Appendix) and that a complex Liouvillian function is a function that is obtained from complex rational functions by a finite process of integrations, exponentiations and algebraic operations.

The proof of Theorem 1 is given in Section 2. In an appendix we have included all the auxiliary results that will be needed to prove Theorem 1.

A singular point is a center if there exists a punctured neighborhood of it filled of periodic orbits, and a focus if it has a punctured neighborhood filled of spiraling orbits. The *center problem* consists in distinguishing between a center or a focus at a monodromic singular point. Placed the singular point at the origin we have a *nondegenerate center* if the linear part has pure imaginary eigenvalues and we have a *nilpotent center* if we have a nilpotent linear part. It is of great importance to know if a system has a finite or infinite number of periodic orbits. The next theorem characterizes the center for system (2).

Theorem 2. *System (2) does not have any center neither nondegenerate nor nilpotent.*

The proof of Theorem 2 is given in Section 3.

Finally, in Section 4 we give an study of the form of the singular first integrals of system (2). A singular first integral around a singular point is a first integral not well-defined in a neighborhood of the singular point. Finding its expansion shows the type of singularity that has a first integral at this singular point.