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Propagation dynamics of a time periodic diffusion equation with degenerate nonlinearity

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This paper is on study of traveling wave solutions and asymptotic spreading of a class of time periodic diffusion equations with degenerate nonlinearity. The asymptotic behavior of traveling wave solutions is investigated by using auxiliary equations and a limit process. In addition, the monotonicity and uniqueness, up to translation, of traveling wave solution with critical speed are determined by sliding method. Finally, combining super and sub-solutions and the stability of steady states, some sufficient conditions on asymptotic spreading are given, which indicates that the success or failure of asymptotic spreading are dependent on the degeneracy of nonlinearity as well as the size of compact support of initial value.

1. Introduction

In this paper, we consider the following time periodic equation

$$u_t(t,x) = u_{xx}(t,x) + f(t,u),\ \ (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{1.1}$$

where $f(t,u)$ satisfies

(F1) $f(t,u) \in C^{\alpha/2,1+\alpha}(\mathbb{R} \times [0,1], \mathbb{R})$ is T-periodic in t with some $\alpha \in (0,1)$ and $T > 0$;
(F2) $f(t,0) = f(t,1) = 0$ with $t \in \mathbb{R}$ and $f(t,u) > 0$ in $\mathbb{R} \times (0,1)$;
(F3) $f_u(t,0) = 0$ and $f_u(t,1) < 0$ for all $t \in \mathbb{R}$, here $f_u(t,0) = \lim_{u\to 0^+} \frac{f(t,u)}{u}$, $f_u(t,1) = \lim_{u\to 1^-} \frac{f(t,u)}{u-1}$.

Several interesting phenomena ranging from population dynamics to chemical waves are modeled by such equations, see [1–3]. In population dynamics, $f_u(t,0) = 0$ is proposed due to the difficulties to find mates and the lack of genetic diversities at small densities, which is called weak Allee effect [4,5]. A typical example

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satisfying (F1)–(F3) is $f(t,u) = r(t)u^p(1-u)$ with $p > 1$ and a T-periodic function $r(t) > 0$. Particularly, (F2) and (F3) imply that $f(t,u)$ is monostable and degenerate.

In what follows, we shall study the traveling wave solutions and spreading speed of (1.1), and we first show the definitions. A traveling wave solution of (1.1) is a special entire solution taking the form $u(t,x) = \phi(t,z), z = x + ct$, in which ϕ is the wave profile propagating through the one-dimensional spatial domain $\mathbb{R}$ at a constant speed c. If we consider the traveling wave solutions of (1.1) connecting 0 with 1, then one has

$$\begin{cases} \phi_t(t,z) = \phi_{zz}(t,z) - c\phi_z(t,z) + f(t,\phi), (t,z) \in \mathbb{R}^2, \\ 0 \le \phi(t,z) \le 1, (t,z) \in \mathbb{R}^2, \\ \phi(t,-\infty) = 0, \phi(t,+\infty) = 1, t \in \mathbb{R}, \\ \phi(t+T,z) = \phi(t,z), (t,z) \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

where $\phi(t,\pm\infty)$ denotes the limit of $\phi(t,z)$ as $z \to \pm\infty$. The spreading speed is an important concept that plays a key role in population dynamics, and it is useful in understanding how the individuals of population spread in a spatial environment, which is defined by the following.

Definition 1.1. Suppose that $u(t,x)$ is a nonnegative function for $(t,x) \in (0,\infty) \times \mathbb{R}$. Then c^* is called a spreading speed of $u(t,x)$ if

(a) $\lim_{t\to+\infty}\sup_{|x|>ct} u(t,x) = 0$ for any given $c > c^*$;
(b) $\liminf_{t\to+\infty}\inf_{|x|<ct} u(t,x) > 0$ for any given $c \in (0,c^*)$.

For the time periodic equation (1.1), the existence of traveling wave solutions and spreading speed have been established in Liang et al. [6] (see also [7]), which can be described by the following lemma.

Lemma 1.2. *Suppose that (F1)–(F3) hold. Then there exists a constant $c^* > 0$ such that (1.2) admits a spatially increasing solution $\phi(t,z) \in C^{1,2}(\mathbb{R}^2,[0,1])$ for all $c \ge c^*$, while no such a solution exists for $c < c^*$. Moreover, let $u(0,x) \in C(\mathbb{R},[0,1])$ be an initial value of (1.1), then the solution $u(t,x)$ of (1.1) is a classical solution in $(0,+\infty) \times \mathbb{R}$ and the following properties hold.*

(a) *For any $c > c^*$, if $u(0,x)$ has non-empty compact support, then*

$$\lim_{t\to+\infty,|x|>ct} u(t,x) = 0.$$

(b) *For any given $\sigma \in (0,1)$, there is a positive number r_σ such that if $u(0,x) > \sigma$ for x on an interval of length $2r_\sigma$, then*

$$\lim_{t\to+\infty,|x|<ct} u(t,x) = 1$$

for any $c \in (0,c^*)$.

Although the asymptotic behavior of solutions of (1.2) with $f_u(t,0) \neq 0$ has been established in [1,8–11], the precise properties of traveling wave solutions (e.g., asymptotic behavior and uniqueness) remain open when $f_u(t,0) = 0$. Moreover, Lemma 1.2 implies that the success of asymptotic spreading depends on the size of support of initial value, which means that the spreading speed is c^* if the size of compact support of initial value is sufficiently large. Then a natural problem is how can the population spread with an initial value having an arbitrary compact support. Furthermore, we want to know whether the degeneracy of nonlinearity effects the spreading.

Based on the results in [6], this paper studies detailed properties of traveling wave solutions and success or failure of asymptotic spreading of (1.1). Throughout most of the above results, the nonzero eigenvalues