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Zero inertia density limit for the hyperbolic system of Ericksen–Leslie's liquid crystal flow with a given velocity

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Formally when the inertia constant goes to zero, the hyperbolic system of Ericksen–Leslie's liquid crystal flow reduces to the corresponding parabolic system. In this paper, under the assumptions that the initial data are well-prepared and the fluid velocity is given, we justify this limit in the context of classical solutions.

1. Introduction

In this paper, we consider the following hyperbolic system

$$\begin{cases} \epsilon D_{\mathbf{u}}^2 \mathbf{d}^\epsilon + D_{\mathbf{u}} \mathbf{d}^\epsilon = \Delta \mathbf{d}^\epsilon + \big(|\nabla \mathbf{d}^\epsilon|^2 - \epsilon |D_{\mathbf{u}} \mathbf{d}^\epsilon|^2 - \lambda_2 \mathbf{d}^{\epsilon\top} \mathbf{A} \mathbf{d}^\epsilon \big) \mathbf{d}^\epsilon + \mathbf{B} \mathbf{d}^\epsilon + \lambda_2 \mathbf{A} \mathbf{d}^\epsilon \,, \\ \mathbf{d}^\epsilon \in \mathbb{S}^2 \,, \end{cases} \tag{1.1}$$

on $\mathbb{R}^+ \times \mathbb{R}^3$ with the initial data

$$\mathbf{d}^\epsilon \big|_{t=0} = \mathbf{d}^{\epsilon,in}(x) \in \mathbb{S}^2 \,,\ D_{\mathbf{u}} \mathbf{d}^\epsilon \big|_{t=0} = \tilde{\mathbf{d}}^{\epsilon,in}(x) \in \mathbb{R}^3 \,, \tag{1.2}$$

where $\mathbf{u} \equiv \mathbf{u}(t,x) \in \mathbb{R}^3$ is a given smooth enough bulk velocity, and $\mathbf{A} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$, $\mathbf{B} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^\top)$, and the initial data obey the compatibility condition $\mathbf{d}^{\epsilon,in} \cdot \tilde{\mathbf{d}}^{\epsilon,in} = 0$. Here the symbol $D_{\mathbf{u}} \mathbf{d}^\epsilon \equiv \partial_t \mathbf{d}^\epsilon + \mathbf{u} \cdot \nabla \mathbf{d}^\epsilon$ represents the material derivative of $\mathbf{d}^\epsilon$ with respect to the given smooth velocity $\mathbf{u}$, and $D_{\mathbf{u}}^2 \mathbf{d}^\epsilon \equiv D_{\mathbf{u}}(D_{\mathbf{u}} \mathbf{d}^\epsilon)$ is the second order material derivative. This system describes the evolution of the direction $\mathbf{d}^\epsilon$ of the liquid molecular on a given background fluid with a fixed bulk velocity $\mathbf{u}$, and the positive constant $\epsilon > 0$ is the inertia density number, which is often small in physical experiment. Consequently, our goal in this paper is

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to rigorously justify the limit from the system (1.1) to the following parabolic flow

$$\begin{cases} D_{\mathbf{u}}\mathbf{d}_0 = \Delta\mathbf{d}_0 + (|\nabla\mathbf{d}_0|^2 - \lambda_2\mathbf{d}_0^\top\mathbf{A}\mathbf{d}_0)\mathbf{d}_0 + \mathbf{B}\mathbf{d}_0 + \lambda_2\mathbf{A}\mathbf{d}_0\,, \\ \mathbf{d}_0 \in \mathbb{S}^2\,, \end{cases} \tag{1.3}$$

on $\mathbb{R}^+\times\mathbb{R}^3$ with the well-prepared initial data.

1.1. Motivations

The system (1.1) can be regarded as a special case of Ericksen–Leslie's parabolic–hyperbolic liquid crystal model for incompressible flow. The hydrodynamic theory of liquid crystals was mathematically established by Ericksen [1–3] and Leslie [4,5] in the 1960s (see also Section 5.1 of [6]). The so-called hyperbolic Ericksen–Leslie's liquid crystal system consists of the following equations:

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{2}\mu_4\Delta\mathbf{u} + \nabla p = -\mathrm{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}) + \mathrm{div}\tilde{\sigma}\,, \\ \mathrm{div}\mathbf{u} = 0\,, \\ \epsilon D_{\mathbf{u}}^2\mathbf{d} = \Delta\mathbf{d} + \gamma\mathbf{d} + \lambda_1(D_{\mathbf{u}}\mathbf{d} - \mathbf{B}\mathbf{d}) + \lambda_2\mathbf{A}\mathbf{d}\,, \end{cases} \tag{1.4}$$

on $\mathbb{R}^+\times\mathbb{R}^3$ with the geometric constraint $|\mathbf{d}|=1$, where γ is Lagrangian multiplier of the form

$$\gamma \equiv \gamma(\mathbf{u},\mathbf{d},D_{\mathbf{u}}\mathbf{d}) = -\epsilon|D_{\mathbf{u}}\mathbf{d}|^2 + |\nabla\mathbf{d}|^2 - \lambda_2\mathbf{d}^\top\mathbf{A}\mathbf{d}\,,$$

and the stress tensor $\tilde{\sigma}$ is

$$\begin{aligned}\tilde{\sigma}_{ij} \equiv \big(\tilde{\sigma}(\mathbf{u},\mathbf{d},D_{\mathbf{u}}\mathbf{d})\big)_{ij} =& \mu_1\mathbf{d}_k\mathbf{d}_p\mathbf{A}_{kp}\mathbf{d}_i\mathbf{d}_j + \mu_2\mathbf{d}_j((D_{\mathbf{u}}\mathbf{d})_i + \mathbf{B}_{ki}\mathbf{d}_k) \\ &+ \mu_3\mathbf{d}_i((D_{\mathbf{u}}\mathbf{d})_j + \mathbf{B}_{kj}\mathbf{d}_k) + \mu_5\mathbf{d}_j\mathbf{d}_k\mathbf{A}_{ki} + \mu_6\mathbf{d}_i\mathbf{d}_k\mathbf{A}_{kj}\,,\end{aligned}$$

where the constants μ_i $(1\le i\le 6)$ are called Leslie coefficients. Moreover, we have the following coefficient relations

$$\lambda_1 = \mu_2 - \mu_3\,, \lambda_2 = \mu_5 - \mu_6\,, \mu_2 + \mu_3 = \mu_6 - \mu_5\,,$$

where the last relation is called *Parodi* relation. For the more background and derivation of (1.4), see [4] and [7].

For any fixed $\epsilon > 0$, in [7] the first two authors of the current paper proved the local well-posedness of the system (1.4) under the assumptions on the Leslie coefficients which ensure the dissipation of the basic energy law, and global well-posedness with small initial data under further damping effect, i.e. $\lambda_1 < 0$. As mentioned in the "Conclusion" section of [7], physically, the inertia constant $\epsilon > 0$ is very small. Formally letting $\epsilon = 0$ will deduce the parabolic Ericksen–Leslie system which is basically a coupling of Navier–Stokes equations and heat flow to the unit sphere. Namely,

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{2}\mu_4\Delta\mathbf{u} + \nabla p = -\mathrm{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}) + \mathrm{div}\tilde{\sigma}\,, \\ \mathrm{div}\mathbf{u} = 0\,, \\ -\lambda_1 D_{\mathbf{u}}\mathbf{d} = \Delta\mathbf{d} + (|\nabla\mathbf{d}|^2 - \lambda_2\mathbf{d}^\top\mathbf{A}\mathbf{d})\mathbf{d} - \lambda_1\mathbf{B}\mathbf{d} + \lambda_2\mathbf{A}\mathbf{d}\,, \end{cases} \tag{1.5}$$

on $\mathbb{R}^+\times\mathbb{R}^3$ with constraint $|\mathbf{d}|=1$. However, deriving the estimates uniform in ϵ of the system (1.4) is very hard, thus justifying the limit as $\epsilon\to 0$ will be highly nontrivial. Even if we adopt the method of Hilbert expansion, the second material derivative $D_{\mathbf{u}}^2\mathbf{d}$ in the third equation of the system (1.4) will deduce a term involving $\partial_t\mathbf{u}_R$ in the remainder system, which will be too hard to be controlled.

Out of consideration of these facts, in the current paper, we consider a simple case, i.e. the bulk velocity $\mathbf{u}$ is given with $\mathrm{div}\mathbf{u}=0$ and sufficiently smooth, and take the coefficient $\lambda_1 = -1$ in (1.4). This case is in fact the system (1.1), which is so-called Ericksen–Leslie's hyperbolic liquid crystal flow with a given bulk velocity. As a consequence, the limit system will formally be (1.3), which can be regarded as Ericksen–Leslie's parabolic liquid crystal flow with a given bulk velocity.