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Fractional Hardy–Sobolev inequalities on half spaces

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A B S T R A C T

We investigate the existence of extremals for Hardy–Sobolev inequalities involving the Dirichlet fractional Laplacian $(-\Delta)^s$ of order $s \in (0,1)$ on half-spaces.

1. Introduction

We study Hardy–Sobolev type inequalities for the restricted Dirichlet fractional Laplacian $(-\Delta)^s$ acting on functions that vanish outside a half-space, for instance outside

$$\mathbb{R}^n_+ = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \; x_1 > 0\}.$$

We always assume $s \in (0,1)$, $n > 2s$ and we put

$$2^*_s := \frac{2n}{n-2s}.$$

We recall that the operator $(-\Delta)^s$ is defined by

$$\mathcal{F}\big[(-\Delta)^s u\big] = |\xi|^{2s}\mathcal{F}[u], \quad u \in \mathcal{C}^\infty_0(\mathbb{R}^n),$$

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where $\mathcal{F}$ is the Fourier transform $\mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx$. The corresponding quadratic form is given by

$$\langle(-\Delta)^s u, u\rangle = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[u]|^2 d\xi.$$

Motivated by applications to variational fractional equations on half-spaces, in the present paper we study the inequality

$$\langle(-\Delta)^s u, u\rangle \ge \lambda \int_{\mathbb{R}^n_+} x_1^{-2s} |u|^2 dx + \mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+) \Big(\int_{\mathbb{R}^n_+} x_1^{-pb} |u|^p dx \Big)^{\frac{2}{p}}, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^n_+) \tag{1.1}$$

under the following hypotheses on the data:

$$2 < p \le 2_s^*, \quad \lambda < \mathcal{H}_s := \frac{1}{\pi} \Gamma\big(s + \frac{1}{2}\big)^2 \tag{1.2a}$$

$$\frac{b}{n} = \frac{1}{p} - \frac{1}{2_s^*}. \tag{1.2b}$$

The bounds on the exponent p are due to Sobolev embeddings; the relation (1.2b) is a necessary condition to have of (1.1) for some constant $\mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+) > 0$, use a rescaling argument.

Actually the assumptions (1.2a)–(1.2b) are sufficient to have that (1.1) holds with a positive best constant $\mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+)$. Here is the argument.

First, notice that for $p = 2_s^*$, that implies $b = 0$, we have

$$\mathcal{S}_s := \inf_{\substack{u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \\ u \ne 0}} \frac{\langle(-\Delta)^s u, u\rangle}{\|u\|_{2_s^*}^2} = \inf_{\substack{u \in \mathcal{C}_0^\infty(\mathbb{R}^n_+) \\ u \ne 0}} \frac{\langle(-\Delta)^s u, u\rangle}{\|u\|_{2_s^*}^2} = \mathcal{S}_s^{0,2_s^*}(\mathbb{R}^n_+) \tag{1.3}$$

because of the action of translations and dilations in $\mathbb{R}^n$. The explicit value of the Sobolev constant $\mathcal{S}_s$ has been computed in [3].

Next, recall the Hardy-type inequality with cylindrical weights proved by Bogdan and Dyda in [2]. It turns out that

$$\langle(-\Delta)^s u, u\rangle \ge \mathcal{H}_s \int_{\mathbb{R}^n_+} x_1^{-2s} u^2 dx \qquad \text{for any } u \in \mathcal{C}_0^\infty(\mathbb{R}^n_+), \tag{1.4}$$

with a sharp constant in the right hand side. Thus $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}^n_+) > 0$ for any $\lambda < \mathcal{H}_s$.

If $p \in (2, 2_s^*)$ and (1.2a)–(1.2b) are satisfied, the existence of a positive constant $\mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+)$ such that (1.1) holds is easily proved via Hölder interpolation between the Sobolev and the cylindrical Hardy inequalities.

We now set up an appropriate functional setting to study the existence of extremals for $\mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+)$. The quadratic form $\langle(-\Delta)^s u, u\rangle$ induces an Hilbertian structure on the space

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{2_s^*}(\mathbb{R}^n) \mid \langle(-\Delta)^s u, u\rangle < \infty\},$$

and $\mathcal{D}^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$ with a continuous embedding by the Sobolev inequality. Clearly $\mathcal{D}^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is the standard Sobolev space $H^s(\mathbb{R}^n)$, see [15] for basic results about H^s-spaces. In particular $\mathcal{D}^s(\mathbb{R}^n) \supsetneq H^s(\mathbb{R}^n)$ and $\mathcal{D}^s(\mathbb{R}^n) \subset H^s_{\rm loc}(\mathbb{R}^n)$, that means $\varphi u \in H^s(\mathbb{R}^n)$ for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}^s(\mathbb{R}^n)$. Therefore, $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}^s(\mathbb{R}^n)$ and the Rellich–Kondrashov Theorem holds, that is, $\mathcal{D}^s(\mathbb{R}^n)$ is compactly embedded into $L^q_{\rm loc}(\mathbb{R}^n)$ for any $q < 2_s^*$.

Next, let $\widetilde{\mathcal{D}}^s(\mathbb{R}^n_+)$ be the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n_+)$ in $\mathcal{D}^s(\mathbb{R}^n)$. We have

$$\widetilde{\mathcal{D}}^s(\mathbb{R}^n_+) = \{u \in \mathcal{D}^s(\mathbb{R}^n) \mid u \equiv 0 \ \text{ on } \ \mathbb{R}^n_- := \mathbb{R}^n \setminus \overline{\mathbb{R}}^n_+\},$$

$$\mathcal{S}_s^{\lambda,p}(\mathbb{R}^n_+) = \inf_{\substack{u \in \widetilde{\mathcal{D}}^s(\mathbb{R}^n_+) \\ u \ne 0}} \frac{\langle(-\Delta)^s u, u\rangle - \lambda \|x_1^{-s} u\|_2^2}{\|x_1^{-b} u\|_p^2}. \tag{1.5}$$