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# Continuity of minimizers to weighted least gradient problems☆

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ABSTRACT

We revisit the question of existence and regularity of minimizers to the weighted least gradient problem with Dirichlet boundary condition

$$\inf\left\{\int_\Omega \mathfrak{a}(x)|Du| : u\in BV(\Omega),\ u|_{\partial\Omega}=g\right\},$$

where $g \in C(\partial\Omega)$, and $\mathfrak{a} \in C^2(\bar{\Omega})$ is a weight function that is bounded away from zero. Under suitable geometric conditions on the domain $\Omega \subset \mathbb{R}^n$, we construct continuous solutions of the above problem for any dimension $n \geq 2$, by extending the Sternberg–Williams–Ziemer technique (Sternberg et al., 1992) to this setting of inhomogeneous variations. We show that the level sets of the constructed minimizer are minimal surfaces in the conformal metric $\mathfrak{a}^{2/(n-1)}I_n$. This result complements the approach in Jerrard et al. (2018) since it provides a continuous solution even in high dimensions where the possibility exists for level sets to develop singularities. The proof relies on an application of a strict maximum principle for sets with area minimizing boundary established by Leon Simon in Simon (1987).


## 1. Introduction

In this article we revisit the question of existence and regularity of solutions in higher dimensions to weighted least gradient problems subject to a Dirichlet boundary condition

$$\inf\left\{\int_\Omega \mathfrak{a}(x)|Du| : u\in BV(\Omega),\ u|_{\partial\Omega}=g\right\}, \tag{1}$$

where $g \in C(\partial\Omega)$, and $\mathfrak{a} \in C^2(\bar{\Omega})$ is a weight function that is bounded away from zero. Existence, comparison and uniqueness results in all dimensions were recently established in [19] over a general class of integrands that includes the present case, and the solution was shown to be continuous in dimensions $n \leq 7$. The restriction on dimension in [19] is due to an appeal to the regularity theory of hypersurfaces minimizing

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parametric elliptic functionals of Almgren, Schoen and Simon [38,37]. The major thrust of this article is to establish such a continuity result for a minimizer of (1) in higher dimensions $n \geq 8$ as well, using a constructive argument along the lines of that used in [43] for the standard case $\mathfrak{a} \equiv 1$.

Going back to the work of Bombieri, De Giorgi and Giusti in [7], extensive studies of functions of least gradient have been carried out in different contexts. The majority of the existing results for least gradient problems study the case of Dirichlet boundary conditions (see for instance [29,19,13,23]). Nonetheless, Neumann and other types of boundary conditions have been explored (cf. [27,35,32]). In the recent years many authors have spent a significant effort to study weighted least gradient problems and further generalizations, due to its various applications to such areas as imaging conductivity problems, reduced models in superconductivity and superfluidity, models for a description of landsliding, and relaxed models in the theory of elasticity and in optimal design, among others. A list of important investigations in these directions can be found in [4,5,13,14,17,19,23,20,26,27,36,18,42–44,28,29,32–35]. In addition, the time dependent notion of total variation flow has proved to be useful in image processing including denoising and restoration, see for example [6,2,3,25,8]. Further generalizations of least gradient problems in the metric space setting have been explored quite recently in [15,21,22].

Let us now introduce the problem more precisely, as well as the main result of this article. Given $n \geq 2$ arbitrary, a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, and a weight function $\mathfrak{a} \in C^2(\bar{\Omega})$ satisfying the following non-degeneracy condition

$$\min_{\bar{\Omega}} \mathfrak{a} \geq \alpha, \tag{2}$$

for some $\alpha \in (0, \infty)$, we deal with the study of minimizers of the weighted $\mathfrak{a}$-variation functional over the set of $BV(\Omega)$ functions that coincide on the boundary with some data $g : \partial\Omega \to \mathbb{R}$ in the sense of $BV$-traces. That is,

$$\inf_{u \in BV_g(\Omega)} \int_\Omega \mathfrak{a}(x)|Du|, \tag{$\mathfrak{a}$ LGP}$$

where the admissible class is defined via

$$BV_g(\Omega) := \left\{ u \in BV(\Omega) : \forall x \in \partial\Omega,\ \lim_{r\to 0} \operatorname*{ess\,sup}_{y\in\Omega\cap|x-y|<r} |u(y) - g(x)| = 0 \right\}. \tag{3}$$

Here $BV(\Omega)$ denotes the class of functions of bounded variation in $\Omega$ (see [12]).

Let us recall the notion of $\mathfrak{a}$-variation of $u \in BV(\Omega)$ induced by the continuous function $\mathfrak{a} : \Omega \to (0, \infty)$, uniformly bounded away from zero. As introduced by Amar and Belletini in [1], the $\mathfrak{a}$-variation of $u \in BV(\Omega)$ in $\Omega$ is given by

$$\int_U \mathfrak{a}(x)|Du| := \sup\left\{ \int_U u \operatorname{div} Y\, dx : Y \in C_c^\infty(U;\mathbb{R}^n),\ |Y(x)| \leq \mathfrak{a}(x)\ \ \forall x \in \Omega \right\}. \tag{4}$$

This corresponds to the definition of $\phi$-variation of $u$ in [1] for the choice of $\phi(x,\xi) = \mathfrak{a}(x)|\xi|$, which is described in terms of the dual norm $\phi^0(x,\xi) := \sup\{\xi \cdot p : \phi(x,p) \leq 1\}$. In (4) we have used the fact that $\phi^0(x,\xi) = |\xi|/\mathfrak{a}(x)$ for such choice of an inhomogeneous, isotropic norm $\phi$. This notion gives rise to a Radon measure on $\mathbb{R}^n$ induced by $u$ that acts on Borel sets via $B \mapsto \int_B \mathfrak{a}(x)|Du|$, called the $\mathfrak{a}$-variation measure of $u$. By analogy, given any Caccioppoli set $E \subset \mathbb{R}^n$ (i.e. set of finite perimeter, see [12]) we can construct an $\mathfrak{a}$-perimeter measure associated with $E$, which is the Radon measure that on any Borel set $B$ assigns the value

$$\mathscr{P}_{\mathfrak{a}}(E,B) := \int_B \mathfrak{a}(x)|D\chi_E|,$$

where $\chi_E$ is the characteristic function of $E$.