Nonlinear Analysis 168 (2018) 27–31

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

# On inclusion relation between weak Morrey spaces and Morrey spaces

Hendra Gunawan[a], Denny Ivanal Hakim[b,*], Eiichi Nakai[c], Yoshihiro Sawano[b]

[a] *Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia*
[b] *Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minami Ohsawa, Hachioji , Tokyo, 192-0397, Japan*
[c] *Department of Mathematics, Ibaraki University, Mito, Ibaraki, 310-8512, Japan*

ARTICLE INFO

*Article history:*
Received 28 August 2017
Accepted 14 November 2017
Communicated by Enzo Mitidieri

*MSC:*
42B25
42B35

*Keywords:*
Morrey spaces
Weak Morrey spaces
Inclusion

ABSTRACT

In this paper, the size of the embedding constant from weak Morrey spaces into Morrey spaces is specified. As a by-product, the difference between weak Morrey spaces and Morrey spaces is clarified.



## 1. Introduction

We specify the size of the embedding constant from weak Morrey spaces into Morrey spaces. Let $1 \le p \le q < \infty$. For a $p$-locally integrable function $f$ on $\mathbb{R}^n$, its Morrey norm is defined by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{q}-\frac{1}{p}} \left( \int_Q |f(y)|^p \, dy \right)^{\frac{1}{p}},$$

where the supremum is taken over all cubes $Q$ having sides parallel to coordinate axis in $\mathbb{R}^n$. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $p$-locally integrable functions $f$ on $\mathbb{R}^n$ for which $\|f\|_{\mathcal{M}^p_q}$ is finite. There is a routine procedure to define the weak space. In fact, the weak Morrey space $w\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all

* Corresponding author.
*E-mail addresses:* hgunawan@math.itb.ac.id (H. Gunawan), dennyivanalhakim@gmail.com (D.I. Hakim), eiichi.nakai.math@vc.ibaraki.ac.jp (E. Nakai), ysawano@tmu.ac.jp (Y. Sawano).

https://doi.org/10.1016/j.na.2017.11.005

measurable functions $f$ for which the quasi-norm

$$\begin{aligned}\|f\|_{w\mathcal{M}^p_q} &:= \sup_{\lambda>0} \lambda \|\chi_{(\lambda,\infty)}(|f(\cdot)|)\|_{\mathcal{M}^p_q} \\ &= \sup_{\lambda>0} \lambda \sup_{Q\subset\mathbb{R}^n} |Q|^{\frac{1}{q}-\frac{1}{p}} \|\chi_Q\, \chi_{(\lambda,\infty)}(|f(\cdot)|)\|_{L^p} \\ &= \sup_{Q\subset\mathbb{R}^n} |Q|^{\frac{1}{q}-\frac{1}{p}} \|\chi_Q f\|_{wL^p} < \infty,\end{aligned}$$

where $wL^p(\mathbb{R}^n)$ is the weak $L^p$ space. Notice that both entities $\|\cdot\|_{\mathcal{M}^p_q}$ and $\|\cdot\|_{w\mathcal{M}^p_q}$ are defined for $0 < p \le q < \infty$ and that for $0 < v < u \le q < \infty$,

$$w\mathcal{M}^u_q \subseteq \mathcal{M}^v_q \subseteq w\mathcal{M}^v_q.$$

In this note, we seek to prove the following theorem, which is an improvement of [2, Theorem 5.1] for the case $\phi(r) = r^{-\frac{n}{q}}$.

**Theorem 1.1.** *Let $0 < v < u \le q < \infty$. Then*

$$\|f\|_{\mathcal{M}^v_q} \le C(u-v)^{-\frac{1}{u}} \|f\|_{w\mathcal{M}^u_q}, \tag{1.1}$$

*where $C = 2^{\frac{1}{v}} u^{\frac{1}{u}}$. Furthermore, for fixed $u$ and $q$, the bound $(u-v)^{-\frac{1}{u}}$ is sharp whenever $v \to u^-$.*

The estimate (1.1) is proved with ease. Let $u > 0$ be fixed. Then for $0 < v < u$ we have

$$\|\chi_Q f\|_{L^v} \le 2^{\frac{1}{v}} u^{\frac{1}{u}} (u-v)^{-\frac{1}{u}} |Q|^{\frac{1}{v}-\frac{1}{u}} \|\chi_Q f\|_{wL^u}$$

for all $f \in wL^u(\mathbb{R}^n)$ and cubes $Q$. As a result, (1.1) follows for $0 < v < u \le q$. The second part of the theorem, namely the sharpness of the bound $(u-v)^{-\frac{1}{u}}$, will be proved in the next section.

As a by-product of Theorem 1.1, we obtain the following result.

**Theorem 1.2.** *Let $0 < p < q < \infty$. Then there exists $g \in w\mathcal{M}^p_q(\mathbb{R}^n)$ such that $g \notin \mathcal{M}^p_q(\mathbb{R}^n)$.*

The importance of Theorem 1.2 is as follows. It is known that the Hardy–Littlewood maximal operator $M$ which is defined by

$$Mf(x) := \sup_{x\in Q\subset\mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, is bounded from $\mathcal{M}^1_q(\mathbb{R}^n)$ to $w\mathcal{M}^1_q(\mathbb{R}^n)$ by [1], but it is not bounded on $\mathcal{M}^1_q(\mathbb{R}^n)$ by [3, Corollary 5.3]. This suggests that there exists $g \in w\mathcal{M}^1_q(\mathbb{R}^n) \setminus \mathcal{M}^1_q(\mathbb{R}^n)$. In this note, we give a constructive proof of Theorem 1.2.

## 2. The proof of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we make a reduction. Let $v < u$. Define $f := g^{\frac{1}{u}}$. Since

$$\|f\|_{w\mathcal{M}^u_q} = \|g\|^{\frac{1}{u}}_{w\mathcal{M}^1_{\frac{q}{u}}} \text{ and } \|f\|_{\mathcal{M}^v_q} = \|g\|^{\frac{1}{u}}_{\mathcal{M}^{\frac{v}{u}}_{\frac{q}{u}}},$$

we see that (1.1) is equivalent to

$$\|g\|_{\mathcal{M}^{\frac{v}{u}}_{\frac{q}{u}}} \le 2^{\frac{u}{v}} \left(1 - \frac{v}{u}\right)^{-1} \|g\|_{w\mathcal{M}^1_{\frac{q}{u}}}. \tag{2.1}$$

Hence, instead of (1.1), we prove (2.1) with $\frac{v}{u} = p$ and $\frac{q}{u} = 2$. Moreover, a passage to the general case can be achieved by modifying the number "4" in the definition of $f_k$ below. Indeed, one may replace "4" by the number $R := 2^{\frac{q}{q-u}}$ (see [5]). Finally, we also assume that $n = 1$; the passage to the higher dimension is easy.