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Total energy of radial mappings

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ABSTRACT

We prove that, the so called total energy functional defined on the class of radial stretchings between annuli attains its minimum on a total energy diffeomorphism between annuli on $\mathbf{R}^n$. This involves a subtle analysis of some special ODE. The result is an extension of the corresponding 2-dimensional case obtained by Iwaniec and Onninen (2009).

1. Introduction

1.1. Total energy

Assume that $h \in \mathscr{W}^{1,n}$ is a homeomorphism between two annuli $\mathbb{A} = A(r, R)$ and $\mathbb{A}_* = A(r_*, R_*)$ of the Euclidean space $\mathbf{R}^n$. Then the total energy of h is defined by Iwaniec and Onninen in [6] by the formula

$$\mathcal{E}[h] = \frac{\alpha}{|\mathbb{A}_*|}\int_{\mathbb{A}} \|Dh\|^n + \frac{\beta}{|\mathbb{A}|}\int_{\mathbb{A}_*} \|Dh^{-1}\|^n,$$

$\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$. The functional

$$h \to \int_{\mathbb{A}} \|Dh\|^n$$

is called the n-energy functional while the functional

$$h \to \int_{\mathbb{A}_*} \|Dh^{-1}\|^n$$

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is called the distortion functional. We define a radial stretching h as a mapping defined by a homeomorphism $H : [r, R] \mapsto [r_*, R_*]$ so that

$$h(x) = H(|x|)\frac{x}{|x|}.$$

In [6], Iwaniec and Onninen showed that the minimum of total energy for $n = 2$ attained by a stretching diffeomorphism of $\mathbb{A}$ onto $\mathbb{A}_*$. One of their key steps was to solve the principal solution of so called equilibrium equation for the radial mappings. It is the following boundary value problem

$$\begin{cases} \ddot{H} = (H - t\dot{H})\dfrac{(\alpha H\dot{H} + \beta t)\dot{H}^2}{(\alpha t\dot{H}^3 + \beta t)tH} \\ H(r) = r_*, \quad H(R) = R_*. \end{cases} \tag{1.1}$$

Namely they proved the following theorem.

Theorem 1.1 (*[6, Theorem 5. 1]*). *Given $R > r > 0$ and $R_* > r_* > 0$ there exists an unique strictly increasing function $H \in \mathscr{C}^\infty[r, R]$ that solves the* Eq. (1.1) *such that $H[r, R] = [r_*, R_*]$.*

Furthermore, in one of their main results ([6, Theorem 1.4]), they proved that the mapping $h(x) = H(|x|)\frac{x}{|x|}$ is the minimizer of the total energy functional (for $n = 2$). In [6, Theorem 1.5], they showed that this result cannot be extended to the Euclidean space $\mathbb{R}^n$ for $n \geq 4$, however the case $n = 3$ remains an open problem.

In this paper, we extend Theorem 1.1 by proving the following theorem.

Theorem 1.2. *Let $n \geq 3$. If $\mathcal{P}(A, A_*)$ is the family of radial mappings with finite total energy, then there is a radial diffeomorphism $h = h_{\lambda_*}$ that minimizes the functional of total energy $\mathcal{E} : \mathcal{P}(A, A_*) \to \mathbf{R}$.*

The total energy is indeed a linear combination of the two operators, the energy functional and distortion functional. However it turns out that to minimize separately those two functionals do not solve the combination problem [6]. The problem of finding a minimizer throughout certain class of homeomorphism has a long history. We want to refer here to some recent papers concerning minimization problem of harmonic Euclidean energy [4,5] and of non-Euclidean energy [10] of homeomorphisms between given annuli on Euclidean plane and on a Riemannian space respectively. Further, for minimization problem of distortion functional, we refer to the papers [1] and [11]. On the generalization of those problem for the spatial annuli and for n-harmonic energy (respectively (ρ, n)) energy, see the papers [7] and [9].

2. The proof of main result

2.1. Hilbert norm of derivatives of the radial stretching and of its inverse

Assume that $h(x) = H(s)\frac{x}{s}$, where $s = |x|$. Let $\mathcal{H}(s) \stackrel{\text{def}}{=} \frac{H(s)}{s}$. Since $\operatorname{grad}(s) = \frac{x}{|x|}$, we obtain

$$Dh(x) = (\mathcal{H}(s))'\frac{x \otimes x}{s} + \mathcal{H}(s)\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. For $x \in \mathbb{A}$, let $T_1 = N = \frac{x}{|x|}$. Further, let $T_2, \ldots, T_n$ be $n - 1$ unit vectors mutually orthogonal and orthogonal to N. Thus

$$\begin{aligned} \|Dh(x)\|^2 = \sum_{i=1}^n |Dh(x)T_i|^2 &= \sum_{i=1}^n \left| (\mathcal{H}(s))'\frac{\langle T_i, x\rangle}{s}x + \mathcal{H}(s)T_i \right|^2 \\ &= (\mathcal{H}'(s))^2 s^2 + n(\mathcal{H}(s))^2 + 2\mathcal{H}(s)\mathcal{H}'(s)s \\ &= \frac{n-1}{s^2}H^2 + \dot{H}^2. \end{aligned}$$