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Space–time continuous limit of random walks with hyperbolic scaling

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ABSTRACT

We consider space–time inhomogeneous one-dimensional random walks which move by $\pm\Delta x$ in each time interval Δt with arbitrary transition probabilities depending on position and time. Unlike Donsker's theorem, we study the continuous limit of the random walks as $\Delta x, \Delta t \to 0$ under hyperbolic scaling $\lambda_1 \geq \Delta t/\Delta x \geq \lambda_0 > 0$ with fixed numbers λ_1 and λ_0. Our aim is to present explicit formulas and estimates of probabilistic quantities which characterize asymptotics of the random walks as $\Delta x, \Delta t \to 0$. This provides elementary proofs of several limit theorems on the random walks. In particular, if transition probabilities satisfy a Lipschitz condition, the random walks converge to solutions of ODEs. This is the law of large numbers. The results here will be foundations of a stochastic and variational approach to finite difference approximation of nonlinear PDEs of hyperbolic types.

1. Introduction

Let $\gamma = \{\gamma^k\}_{k=0,1,2,\ldots}$, $\gamma^0 = 0$ be the one-dimensional random walk on the rescaled space $\Delta x\mathbb{Z} := \{x_m := m\Delta x \mid m \in \mathbb{Z}\}$, $\Delta x > 0$ defined by the transition probability $\rho(\gamma^k = x_m; \gamma^{k+1} = x_m \pm \Delta x) = 1/2$ and $w_\Delta = \{w_\Delta(t)\}_{t\geq 0}$ be the stochastic process given by the linear interpolation of γ between each $[t_k, t_k + \Delta t]$, where $t_k := k\Delta t \in \Delta t\mathbb{Z}_{\geq 0}$, $\Delta t > 0$. It is easy to check that, as $\Delta := (\Delta x, \Delta t) \to 0$ under the condition $\Delta t/\Delta x \equiv 1$, the distribution of w_Δ converges weakly to the δ-measure supported by $w_0(t) \equiv 0$, or equivalently w_Δ converges to w_0 locally uniformly in probability.

In this paper we study the space–time continuous limit of space–time inhomogeneous random walks as $\Delta \to 0$ *under hyperbolic scaling* $\lambda_1 \geq \Delta t/\Delta x = \lambda \geq \lambda_0 > 0$ *with fixed numbers* λ_1 *and* λ_0. We deal with the random walks $\gamma = \{\gamma^k\}_{k=0,1,2,\ldots}$, $\gamma^0 = 0$ defined by the following transition probabilities which are allowed to be far from a homogeneous one:

$$\rho(\gamma^k = x_m; \gamma^{k+1} = x_m \pm \Delta x) := \frac{1}{2} \pm \frac{1}{2}\lambda\xi(t_k, x_m),$$

where $\xi : (\mathbb{R}_{\geq 0}) \times (\mathbb{R}) \to [-\lambda^{-1}, \lambda^{-1}]$ is a deterministically given function. For simplicity we restrict our arguments to a fixed time interval $[0, T]$ with an arbitrary $T > 0$ and therefore we study the random walks for $0 \leq k \leq K$, where K is a natural number such that $t_K \in (T - \Delta t, T]$ and $K \to \infty$ as $\Delta \to 0$. Let Ω_Δ be the set of all the sample paths of a random walk γ. We have the probability measure on Ω_Δ which is derived from γ. We still use the notation γ for each element of Ω_Δ. We also deal with the random variable $\eta(\gamma)$ on Ω_Δ defined by

$$\eta(\gamma) : \{0, 1, 2, \ldots, K\} \to \mathbb{R}, \qquad \eta^k(\gamma) := \sum_{0\leq k'<k} \xi(t_{k'}, \gamma^{k'})\Delta t, \qquad \eta^0(\gamma) = 0.$$

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Let $\mathcal{W}$ be the set of all continuous functions $f : [0, T] \to \mathbb{R}$ with the C^0-norm. We introduce the stochastic processes $w_\Delta, \tilde{w}_\Delta : \Omega_\Delta \to \mathcal{W}$ which are the linear interpolations of $\gamma, \eta(\gamma)$ respectively. We remark that all the sample paths of w_Δ and $\tilde{w}_\Delta$ are Lipschitz with a common Lipschitz constant (less than λ_0^{-1}) independent of Δ and ξ. The probability measures on $\mathcal{W}$ associated with w_Δ and $\tilde{w}_\Delta$ are denoted by $P_\Delta = P_\Delta(\cdot; \xi)$ and $\tilde{P}_\Delta = \tilde{P}_\Delta(\cdot; \xi)$.

Our aim is to present explicit formulas and estimates of probabilistic quantities of Ω_Δ, which are stated in Section 3. These formulas and estimates lead to possible applications stated in Section 2. They also imply the following basic limit theorems on the asymptotics of $P_\Delta = P_\Delta(\cdot; \xi)$ and $\tilde{P}_\Delta = \tilde{P}_\Delta(\cdot; \xi)$ as $\Delta \to 0$ under $\lambda_1 \geq \lambda = \Delta t/\Delta x \geq \lambda_0 > 0$. The results which hold for any ξ and therefore for any transition probabilities are the following:

Theorem 1.1. 1. *For each uniformly continuous function $\mathcal{L} : \mathcal{W} \to \mathbb{R}$, there exists a number $\varepsilon(\Delta, \mathcal{L}) > 0$ which is independent of ξ and tends to 0 as $\Delta \to 0$ such that*

$$\left| \int_{\mathcal{W}} \mathcal{L}(f) P_\Delta(df) - \int_{\mathcal{W}} \mathcal{L}(f) \tilde{P}_\Delta(df) \right| \leq \varepsilon(\Delta, \mathcal{L}).$$

2. *For each sequence ξ_j, which is not necessarily convergent, and $\Delta_j \to 0$, the sets of probability measures $\{P_{\Delta_j}(\cdot; \xi_j)\}_j$ and $\{\tilde{P}_{\Delta_j}(\cdot; \xi_j)\}_j$ are relatively compact.*

Next we impose a Δ-independent Lipschitz condition on ξ. Then we have the law of large numbers:

Theorem 1.2. *Consider a sequence of continuous functions $\xi_\Delta(t, x) : [0, T] \times [-\frac{T}{\lambda_0}, \frac{T}{\lambda_0}] \to [-\lambda_1^{-1}, \lambda_1^{-1}]$ which is Lipschitz with respect to x with a Lipschitz constant θ independent of Δ and converges uniformly to ξ_0 as $\Delta \to 0$. Let w_0 be the solution of the ODE $w_0'(t) = \xi_0(t, w_0(t)),\ w_0(t) = 0$. Then, for $\xi := \xi_\Delta$ with each fixed Δ, it holds that w_Δ and $\tilde{w}_\Delta$ converge to w_0 uniformly in probability as $\Delta \to 0$, or equivalently P_Δ and $\tilde{P}_\Delta$ converge to δ_{w_0} weakly as $\Delta \to 0$, where δ_{w_0} is the probability measure on $\mathcal{W}$ supported by $\{w_0\}$.*

We remark that some of our argument is a direct and simpler approach to the results in [1, Chapter 8]: it is shown that rescaled continuous-time Markov chains of a certain class defined with Poisson processes converge to solutions of the corresponding ODEs as the scaling parameter goes to infinity. In fact, discrete-time Markov chains can be embedded nontrivially in continuous-time Markov chains of that class (see e.g. [2]) and Theorem 1.2 can be obtained in an advanced setting of probability theories. However we still need the various estimates directly derived from our random walks, which seem to be dim in the continuous-time setting, in order to apply our results to numerical computations of PDEs stated in Section 2.

2. Motivation

There are many ideas of exploiting limit theorems for random variables in analysis of (deterministic or stochastic) differential equations. Our idea is one of them based on the law of large numbers.

The motivation comes from finite difference approximation of nonlinear PDEs. Now we roughly describe stochastic and variational approaches to hyperbolic conservation laws and Hamilton–Jacobi equations, from which our problem naturally arises. Consider

$$u_t + H(t, x, u)_x = 0 \quad \text{in } (0, T] \times \mathbb{R}, \qquad u(0, x) = u_0(x) \quad \text{on } \mathbb{R}, \tag{2.1}$$

$$v_t + H(t, x, v_x) = 0 \quad \text{in } (0, T] \times \mathbb{R}, \qquad v(0, x) = v_0(x) \quad \text{on } \mathbb{R}. \tag{2.2}$$

The solutions of (2.1) and (2.2) necessarily lose their regularity within a finite time interval in general, even if initial data are analytic. Hence (2.1) and (2.2) are analyzed in the classes of generalized solutions called entropy solutions and viscosity solutions respectively. We remark that these two classes are equivalent for one-dimensional problems, namely $u = v_x$, if $u_0 = v_{0x}$. From now on we suppose that $u_0 = v_{0x}$. It is sometimes convenient to deal with (2.1) through (2.2). One of the central achievements in the large literature on (2.1) and (2.2) is that these PDEs can be closely related to deterministic calculus of variations. Under several assumptions, the value of the viscosity solution v at each point (t, x) is given by

$$v(t, x) = \inf_{w \in AC, w(t) = x} \left\{ \int_0^t L(s, w(s), w'(s)) ds + v_0(w(0)) \right\}, \tag{2.3}$$

where AC is the set of absolutely continuous curves and $L(t, x, \cdot)$ is the Legendre transform of $H(t, x, \cdot)$. If v is differentiable in x at (t, x) (this holds for a.e. points), then there exists the unique minimizing curve w_* of (2.3) and the value of the entropy solution u at the (t, x) is given by

$$u(t, x) = \int_0^t L_x(s, w_*(s), w_*'(s)) ds + u_0(w_*(0)). \tag{2.4}$$

The variational approach to (2.1) and (2.2) yields much information on properties of the solutions. Furthermore this approach also contributes to many other fields: the above relations enable us to combine the analysis of PDEs and that of optimal