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The axes of random infinitesimal rotations and the propagation of orientation uncertainty

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ABSTRACT

Perception systems can measure the orientation of a solid 3D object; however, their measurements will contain some uncertainties. In many robotic applications, it is important to propagate the orientation uncertainties of a rigid object onto the uncertainties of specific points on its surface. The orientation uncertainty can be reported as a 3×3 covariance matrix. We show that the off-diagonal elements of this matrix provide important clues about the angular uncertainties of points on the object's surface. Specifically, large off-diagonal elements correspond to a highly concentrated distribution of axes of random infinitesimal rotations which causes large variability in the angular uncertainties of surface points. In particular, experimental data indicate that the ratio of maximum to minimum angular uncertainties can exceed three. In contrast, small off-diagonal elements correspond to a uniform distribution of axes which causes the angular uncertainty of all points on the object's surface to be almost constant.

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1. Introduction

In many robotic applications it is important to understand the uncertainty of specific points on the surface of a rigid body. For example, when planning a safe path of a solid object, the specific points are those that may come close to an obstacle; or when planning a good grip, the specific points are those that may come in contact with a gripper's fingers. Typically, the calculation of these specific points is dependent on noisy orientation data obtained with a pose measuring system. We are interested in characterizing the propagation of the uncertainty of this noisy orientation data onto specific points on the surface of a rigid body.

Typically, in robotic applications the propagation of orientation uncertainty is defined as the propagation along different joints (encoders) on a kinematic chain, or in the context of dynamic control the propagation of orientation uncertainty is defined when the orientation at time $k+1$ depends on noisy orientation measured at time k [1,2]. In contrast, here we are interested in the static configuration where a fixed orientation of a rigid body is repeatedly measured in the same experimental conditions. Mathematically, this can be modeled by studying the uncertainty of

$$\boldsymbol{w}_j = \boldsymbol{R}_j \boldsymbol{u} \quad (1)$$

where $\boldsymbol{R}_j$ is a 3×3 rotation matrix representing the noisy orientation data obtained from a pose measuring system at the j-th measurement and $\boldsymbol{u}$ is a 3D vector representing a specific point on the surface of a rigid body. We are interested in characterizing how the propagated uncertainty changes among different points ($\boldsymbol{u}$) on the surface of the rigid body.

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For this paper, we will make some assumptions on the orientation $\boldsymbol{R}_j$ and point $\boldsymbol{u}$. In particular, we assume that the measured orientation can be broken down as

$$\boldsymbol{R}_j = \boldsymbol{R}_0 \Delta \boldsymbol{R}_j \quad (2)$$

where $\boldsymbol{R}_0$ is the true and usually unknown rotation and $\Delta\boldsymbol{R}_j$ is a random infinitesimal rotation representing the uncertainty of the orientation information at the j-th measurement [3]. We will approximate the unknown true rotation $\boldsymbol{R}_0$ with the average rotation $\boldsymbol{R}_{\mathrm{avg}}$ calculated from averaging many $\boldsymbol{R}_j$ (j = 1,2,…N) that are measured in the same experimental conditions as outlined in *Guide to the Expression of Uncertainty* (GUM) [4]. We should note, however, that there are different ways to calculate $\boldsymbol{R}_{\mathrm{avg}}$ [5].

From (2), we can see that the statistical properties of the measured rotations $\boldsymbol{R}_j$ are defined by the characteristics of the random infinitesimal rotations $\Delta\boldsymbol{R}_j$. In the axis-angle representation, an infinitesimal rotation is defined by a small angle of rotation ρ_j such that $\sin\rho_j \approx \rho_j$ and $\cos\rho_j \approx 1$, regardless of the direction of the corresponding axis. This mathematical property of the axis-angle representation may tempt one to conclude that the spatial distribution of the axis of rotation is irrelevant and does not have any important practical implications [6]. However, in this paper we will show that the axis of a random infinitesimal rotation is important in understanding the propagation of uncertainty from the orientation information of a rigid body onto a specific point on its surface.

Another aspect of the uncertainty propagation deals with linear scaling: points that are further away from the center of rotation (and are not on the axis of rotation) will have larger linear uncertainty compared to points that are closer to the center of rotation. As a result, we will only deal with points $\boldsymbol{u}$ that lie on the unit sphere. Therefore, $\boldsymbol{u}$ can be parameterized by two angles – elevation ϑ and azimuth φ such that $\boldsymbol{u} = \boldsymbol{u}(\vartheta, \varphi)$. Since rotation does not change the length of a vector, the rotated vectors $\boldsymbol{w}_j$ can also be parameterized by two angles – elevation λ and azimuth τ – such that $\boldsymbol{w}_j = \boldsymbol{w}(\lambda_j, \tau_j)$. Thus, we are interested in propagating error from the measured orientation $\boldsymbol{R}_j$ onto the pair of angles λ_j and τ_j. We systematically investigate how the angular uncertainty of the corresponding $\boldsymbol{w}_j$ depends on the covariance matrix of the three parameters defining the random infinitesimal rotations $\Delta\boldsymbol{R}_j$ and the vector $\boldsymbol{u}$ defining a point on the surface of a rigid body. Analysis of experimental data show that, though it may appear paradoxical, the angular uncertainty of different points on the surface of a rotated rigid body may vary up to a factor of three – even for a perfectly symmetrical sphere. We will show that this variability is strongly correlated with the distribution of the axes of the random infinitesimal rotations. Specifically, the variability will be large unless the axes of the random infinitesimal rotations are distributed uniformly about the unit sphere. This distribution of the axes can be determined by inspecting the off-diagonal elements of the covariance matrix of the three parameters defining the random infinitesimal rotations $\Delta\boldsymbol{R}_j$. For large off-diagonal elements, the eigenvector of the covariance matrix corresponding to the smallest eigenvalue represents the point on the unit sphere with the largest uncertainty, while vectors that are perpendicular to this eigenvector represent the points where the smallest angular uncertainty is observed. Coincidentally, these perpendicular vectors also correspond to the region with the highest concentration of axes from the random infinitesimal rotations.

We will organize this paper in the following manner: Section 2 formulates the problem more precisely and reviews the necessary equations, Section 3 describes the experimental setup for obtaining the measured orientations $\boldsymbol{R}_j$ and subsequent data processing, Section 4 contains the results of propagating the uncertainty from the measured orientations $\boldsymbol{R}_j$ onto $\boldsymbol{w}_j$, Section 5 contains a discussion of the consequences of these results, and Section 6 contains concluding remarks.

2. Angular uncertainty of a unit vector

If the Cartesian coordinates of a point $\boldsymbol{w}_j(x,y,z)$ have a Gaussian distribution with the constraint that $||\boldsymbol{w}_j|| = 1$, then the probability of finding a vector $\boldsymbol{w}_j$ on the unit sphere is described by the Fisher–Bingham–Kent (FBK) distribution [7]. FBK distributions are the core of directional statistics [8–10] and were first used in paleomagnetism to study the spatial distribution of magnetic properties in rocks [11]. In applications that are pertinent to perception in robotics, the FBK has recently been used to improve the alignment method which is based on calculating the normal and principal curvature directions on the surface of an object [12,13]. FBK can be used to find the probability of measuring a specific deviation angle μ_j between a unit vector $\boldsymbol{w}_j$ and the vector of average direction $\boldsymbol{w}_{avg}$. The distribution of these angles μ_j is described by two parameters: the angular uncertainty σ and the eccentricity β. Here σ (or equivalent concentration $\kappa = \sigma^{-2}$) describes the spread of $\boldsymbol{w}_j$ around $\boldsymbol{w}_{avg}$ such that a smaller σ indicates a higher (tighter) concentration around $\boldsymbol{w}_{avg}$, and β describes the shape of the elliptical contour of a constant probability on the unit sphere: when $\beta = 0$ the contour is a circle centered at $\boldsymbol{w}_{avg}$ and larger β corresponds to a flatter contour.

While paleomagnetic data may be very noisy and their associated pdf may have large spread, modern pose measuring instruments can provide angular data with uncertainty σ on the order of milliradians κ. For such small angular uncertainties σ, the angle μ_j will be small and the following approximation holds: $\cos\mu_j \approx 1 - 0.5\mu_j^2$ and $\sin\mu_j \approx \mu_j$. Using this approximation, the probability of measuring a specific angle μ can be calculated as the FBK

$$G_{\sigma,\beta}(\mu)d\mu = \mu \exp(-0.5\mu^2/\sigma^2) E_{\sigma,\beta}(\mu) d\mu \quad (3)$$

where $E_{\sigma,\beta}(\mu)$ is the Kent correction to the Fisher distribution due to the nonzero parameter β along the azimuth angle η around $\boldsymbol{w}_{avg}$

$$E_{\sigma,\beta}(\mu) = \frac{1}{2\pi}\sigma^{-2}\sqrt{(1-2\beta\sigma^2)(1+2\beta\sigma^2)} \times \int_0^{2\pi} \exp(\beta\mu^2 \cos 2\eta) d\eta. \quad (4)$$