Games and Economic Behavior ••• (••••) •••–•••

Contents lists available at ScienceDirect

Games and Economic Behavior

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Is Shapley cost sharing optimal? ☆

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A R T I C L E   I N F O

Article history:
Received 29 August 2016
Available online xxxx

JEL classification:
D82

Keywords:
Shapley value
Cost-sharing mechanisms
Approximate efficiency

A B S T R A C T

A general approach to the design of budget-balanced cost-sharing mechanisms is to use the Shapley value, applied to the given cost function, to define payments from the players to the mechanism. Is the corresponding Shapley value mechanism "optimal" in some sense? We consider the objective of minimizing worst-case inefficiency subject to a revenue constraint, and prove results in three different regimes. First, for the public excludable good problem, the Shapley value mechanism minimizes the worst-case efficiency loss over all truthful, deterministic, and budget-balanced mechanisms that satisfy equal treatment. Second, even with randomization and approximate budget-balance allowed and dropping equal treatment, the worst-case efficiency loss of the Shapley value mechanism is within a constant factor of the minimum possible. Third, for no-deficit mechanisms, we prove a general positive result: for every monotone cost function, a suitable blend of the VCG and Shapley value mechanisms is no-deficit and enjoys good approximate efficiency guarantees.

1. Introduction

In a cost-sharing mechanism design problem, several participants with unknown preferences vie to receive some good or service, and each possible outcome has a known cost. Formally, we consider problems defined by a set U of players and a cost function $C : 2^U \to \mathbb{R}^+$ that describes the cost incurred by the mechanism as a function of the outcome (i.e., of the set S of "winners"). We assume that each player i has a private nonnegative value v_i for winning.

☆ The results in Sections 3–5 appeared, in preliminary form, in the extended abstract (Dobzinski et al., 2008). The results in Section 6 appeared in the PhD thesis of the fourth author (Sundararajan, 2009).

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[1] This work was done while the author was visiting Stanford University, and supported by the Adams Fellowship Program of the Israel Academy of Sciences and Humanities, and by grants from the Israel Science Foundation and the United States–Israel Binational Science Foundation.
[2] This work was supported in part by NSF awards CCF-0448664 and CCF-1524062, an ONR Young Investigator Award, and an Alfred P. Sloan Fellowship.
[3] This work was done while a PhD student at Stanford University and supported by NSF Award CCF-0448664 and a Stanford Graduate Fellowship.

http://dx.doi.org/10.1016/j.geb.2017.03.008
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For example, in the *public excludable good* problem (e.g. Deb and Razzolini, 1999b; Moulin and Shenker, 2001), the problem is to determine whether or not to finance a public good and, if so, who is allowed to use it.[4] This problem corresponds to the cost function C with $C(\emptyset) = 0$ and $C(S) = 1$ for every $S \neq \emptyset$. Many other cost functions have been considered in the cost-sharing literature (Section 1.2), and most of them include public excludable good problems as a special case.

A (direct-revelation) *cost-sharing mechanism* is a protocol that decides, as a function of players' bids, which players win and at what prices. For example, the general Vickrey–Clarke–Groves (VCG) mechanism specializes to the following procedure for a public excludable good problem.

VCG mechanism (public excludable good)

1. Accept a bid b_i from each player i.
2. Choose the outcome $S := U$ if $\sum_{i \in U} b_i > 1$, and $S := \emptyset$ otherwise.
3. Charge each winner i the minimum bid for which she would still win (holding others' bids fixed), namely $\max\{0, 1 - \sum_{j \in U \setminus \{i\}} b_j\}$.

It is well known that the VCG mechanism is *truthful*, meaning that for every player it is a dominant strategy to set her bid equal to her private value for winning. By design, the VCG mechanism is also *efficient*, meaning that it always selects the set $S \subseteq U$ of winners that maximizes the total value to the winners less the cost incurred, that is, the *social welfare* $\sum_{i \in S} v_i - C(S)$. One drawback of the VCG mechanism is that its revenue can be far from the cost incurred. For example, in a public excludable good problem in which all of the players have valuations larger than $\frac{1}{|U|-1}$, the VCG mechanism obtains zero revenue (while the cost is 1).

A second approach to designing a cost-sharing mechanism is to insist on *budget balance*, meaning that the sum of players' payments equals the cost of the outcome chosen. For a symmetric problem like a public excludable good problem, perhaps the most natural approach is to require equal payments from the winners, and subject to this choose as many winners as possible. The Shapley value mechanism implements this idea. For the special case of a public excludable good, the Shapley value mechanism chooses the largest set S of players such that $b_j \geq 1/|S|$ for all $j \in S$. The mechanism can be described more procedurally as follows.[5]

Shapley value mechanism (public excludable good)

1. Accept a bid b_i from each player i.
2. Initialize $S := U$.
3. If $b_i \geq 1/|S|$ for every $i \in S$, then halt with winners S, and charge each player $i \in S$ the price $p_i = 1/|S|$.
4. Let $i^* \in S$ be a player with $b_{i^*} < 1/|S|$.
5. Set $S := S \setminus \{i^*\}$ and return to Step 3.

The Shapley value mechanism is also truthful—overbidding can only cause a player to win when she would prefer to lose, and vice versa for underbidding. By design, it is budget-balanced. It is not efficient, however.

Example 1.1 *(Inefficiency of Shapley value mechanism).* Consider a public excludable good problem with k players, where the valuation of player i is $\frac{1}{i} - \delta$ for small $\delta > 0$. By induction, the Shapley value mechanism will remove player $k + 1 - i$ in its ith iteration, terminating with the empty outcome, which has zero social welfare. The welfare-maximizing outcome is to choose the full set $S = U$ of winners. This results in social welfare approaching $\mathcal{H}_k - 1$ as $\delta \to 0$, where $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i}$ denotes the kth Harmonic number (which lies between $\ln k$ and $\ln k + 1$).

Thus, the VCG mechanism sacrifices budget-balance in the service of efficiency, while the Shapley value mechanism makes the opposite trade-off. This trade-off between efficiency and budget-balance is fundamental: no truthful mechanism can be both (Green et al., 1976; Roberts, 1979). This impossibility result raises the issue of understanding the feasible trade-offs between the two objectives.

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[4] In 1959, the citizens of Palo Alto, Portola Valley, and Los Altos Hills voted on whether or not to finance a new park. The measure only passed in Palo Alto, and to this day entrance to Foothills Park is restricted to Palo Alto residents.

[5] We call this mechanism the Shapley value mechanism (following Moulin and Shenker, 2001) because the prices charged to the winning set S correspond to the Shapley value applied to the cost function C restricted to S (since C is symmetric, the Shapley values are equal). The Shapley value mechanism can be defined analogously for arbitrary cost-sharing problems (see Moulin and Shenker, 2001).

Please cite this article in press as: Dobzinski, S., et al. Is Shapley cost sharing optimal?. Games Econ. Behav. (2017), http://dx.doi.org/10.1016/j.geb.2017.03.008