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# Estimation error in mean returns and the mean-variance efficient frontier☆

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ABSTRACT

In this paper, we build estimation error in mean returns into the mean-variance (MV) portfolio theory under the assumption that returns on individual assets follow a joint normal distribution. We derive the conditional sampling distribution of the MV portfolio along with its mean and risk return when the sample covariance matrix is equal to the population covariance matrix. We use the mean squared error (MSE) to characterize the effects of estimation error in mean returns on the joint sampling distributions and examine how such error affects the risk-return tradeoff of the MV portfolios. We show that the negative effects of error in mean returns on the joint sampling distributions increase with the decision maker's risk tolerance and the number of assets in a portfolio, but decrease with the sample size.

## 1. Introduction

In Markowitz's (1952) paradigm, known as the mean-variance (MV henceforth) model, the objective of a portfolio decision maker (DM henceforth) is to choose a portfolio on the efficient set under the assumption that she has the perfect information on the model parameters – the expected returns on individual assets and the corresponding covariance matrix. In the real world, however, the DM has to estimate the parameters using historical data. Numerous studies, (Best & Grauer, 1991; Broadie, 1993; Chopra & Ziemba, 1993; Litterman, 2004; Michaud, 1989), have shown that bad estimates based on the historical approach that arise from estimation errors in the moments of return distributions can render inferior performance ex-post.

Researchers have striven to derive robust estimates for the MV model to mitigate the negative impacts of estimation errors on portfolio performance. Since Merton (1980), many researchers have shifted their efforts to the global minimum variance (GMV henceforth) portfolio, whose weights depend solely on the more stable covariance matrix. For example, Jagannathan and Ma (2003) and DeMiguel, Garlappi, and Uppal (2009) show that the GMV portfolio outperforms portfolios that require estimating mean returns. Motivated by these findings, researchers have proposed alternative approaches to robust covariance matrix estimates. For instance, Ledoit and Wolf (2003) develop a shrinkage approach that generates a lower GMV portfolio variance than the conventional sample covariance matrix; Candelon, Hurlin, and Tokpavi (2012) provide a double shrinkage approach and show based on Monte Carlo simulation that the double shrinkage approach can be more beneficial when the estimation window is small.

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While many past studies have focused on the GMV portfolio and developed robust estimates of the covariance matrix, it is still important and fruitful to solve for a mean-variance optimal portfolio should the DM have a reliable approach to tackling estimation errors in mean returns.[1] For example, Black and Litterman (1992) demonstrate that a small change in the vector of mean returns can cause large variation in portfolio positions. DeMiguel and Nogales (2009) illustrate how fluctuations in mean returns affect both the mean-variance portfolio and the GMV portfolio, and show that the effect on the former is much more pronounced than on the latter.

In this paper, we provide a theoretical framework that builds estimation error in mean returns into the MV paradigm while assuming that asset returns follow a joint normal distribution. Our main results are derived for the conditional case when the sample covariance matrix is equal to the population covariance matrix. We consider the conditional case for two main reasons. First, we are motivated by previous findings that the effect of estimation error in the covariance matrix on portfolio performance is much less severe than the mean returns (see e.g., Chopra and Ziemba (1993)). Second, we trade-off between accuracy and perspicacity. The results for the conditional case are less complicated than the unconditional one. This, therefore, allows us to inspect the implications of estimation errors for portfolio performance in a more tractable manner.

We derive the conditional joint sampling distributions of the weights of the MV portfolio, along with its mean and risk return when the sample covariance matrix is equal to the population covariance matrix. Using the mean squared error (MSE henceforth) to characterize estimation error, we show analytically and numerically that estimation errors in the weights of the MV portfolio and its corresponding mean and variance increase with the level of risk tolerance and the number of assets under consideration, while it decreases with the sample size. These results suggest that when there is a small number of assets in the portfolio, the benefit of diversification outweighs the cost arising from estimation errors. However, as we keep adding more assets to the portfolio, the cost will eventually dominate the benefit.

We are further interested in the effects of estimation error in mean returns on the risk-return tradeoff of efficient portfolios. In the conventional MV model, the tradeoff is deterministic and given by a parabola that defines a set of optimal portfolios, known as the mean-variance efficient frontier (MVE henceforth). This relation, however, does not hold in practice, due to the presence of estimation error, (Cochrane, 2014). In the paper, we show that for every efficient portfolio on the classical MVE frontier there is a joint distribution between the mean and variance of the MV portfolio return.

Our paper is related to the work by Bodnar and Schmid (2009a). The authors derive the marginal sampling distribution for the mean and variance of each portfolio when the expected returns and the covariance matrix of individual assets are both unknown. Different from Bodnar and Schmid (2009a), we explore the implications of estimation errors for the MVE frontier. We do so graphically with emphasis on the estimation error related factors, i.e. sample size, number of assets, and risk tolerance. Okhrin and Schmid (2006), another related paper, also examine what affects estimation errors in the MV portfolio weights when both the expected returns and the covariance matrix of individual assets are both unknown. However, they do not cover the estimation error effect on the mean and risk return of the MV portfolio.

Our paper is also related to Bodnar and Schmid (2011). The authors derive the distribution of a linear combination of the estimated portfolio weights that allows the decision maker to characterize the distribution of specific positions in the portfolio, which we refer to as the unconditional case. We focus on a conditional case in which the sample covariance matrix is equal to the population covariance matrix. In the paper, we explore the differential implications of Bodnar and Schmid's (2011) and our model. Our numerical exercises suggest that the discrepancy between the two models decreases as the sample size increases. Consistent with past studies, this result indicates that estimation error induced by the mean vector plays a more important role than that induced by the covariance matrix.

The remainder of this paper is organized as follows. Section 2 provides an algebraic review of the traditional mean-variance portfolio theory, in which we do not consider estimation errors in the portfolio analysis. In Section 3, we incorporate estimation error into the portfolio problem and summarize our main findings. Section 4 provides robustness check. Section 5 concludes. We present our main findings in four propositions and report proofs in the appendix.

## 2. Review of asset allocation

In this section, we provide a short algebraic review of asset allocation theory and a number of important properties of the MV efficient set. While this review assumes full information and omits estimation error, the analytical framework serves as the building block of our analysis in Section 3.

Let $\mathbf{r} = (r_1, \ldots, r_d)\prime$ be the vector of returns on $d$ assets. We assume that returns are independent over time and follow a joint normal distribution, such that $\mathbf{r} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are, respectively, the mean vector and the covariance matrix of the asset returns. The vector of asset weights is denoted by $\boldsymbol{\xi} = (\xi_1, \ldots \xi_d)$, where $\xi_i$ represents the proportion of wealth allocated in asset $i$, $\forall i = 1, \ldots, d$. The investor is assumed to hold an initial wealth of one dollar that she allocates among the $d$ risky assets, such that $\boldsymbol{\xi}\prime\mathbf{e} = 1$, where $\mathbf{e}$ is a $d \times 1$ vector of ones. The portfolio return is $\mathbf{r}_p = \boldsymbol{\xi}\prime\mathbf{r}$ and follows a univariate normal distribution with a mean and variance equal to $\boldsymbol{\eta}_p = \boldsymbol{\xi}\prime\boldsymbol{\mu}$ and $\boldsymbol{\sigma}_p^2 = \boldsymbol{\xi}\prime\boldsymbol{\Sigma}\boldsymbol{\xi}$, respectively.

The DM with a risk aversion of $\kappa$ chooses her optimal portfolio by solving the following optimization problem:

$$\max_{\boldsymbol{\xi}} \boldsymbol{\xi}\prime\boldsymbol{\mu} - 0.5\kappa\boldsymbol{\xi}\prime\boldsymbol{\Sigma}\boldsymbol{\xi} \text{s.t.} \boldsymbol{\xi}\prime\mathbf{e} = 1 \tag{2.1}$$

The solution of (2.1) yields

[1] For instance, Zhu (2013) and DeMiguel, Nogales, and Uppal (2014) investigate assets returns predictability and its implications for optimal portfolio selection.