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Analytic uncertainty and sensitivity analysis of models with input correlations

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HIGHLIGHTS

- An analytic expression is derived for variance propagation with input correlations.
- Coefficients of independent and correlated parts of any variables are specified.
- Independent, correlated and coupling variance contributions are defined.
- A practical application is proposed to the analysis of a deterministic HIV model.

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ABSTRACT

Probabilistic uncertainty analysis is a common means of evaluating mathematical models. In mathematical modeling, the uncertainty in input variables is specified through distribution laws. Its contribution to the uncertainty in model response is usually analyzed by assuming that input variables are independent of each other. However, correlated parameters are often happened in practical applications. In the present paper, an analytic method is built for the uncertainty and sensitivity analysis of models in the presence of input correlations. With the method, it is straightforward to identify the importance of the independence and correlations of input variables in determining the model response. This allows one to decide whether or not the input correlations should be considered in practice. Numerical examples suggest the effectiveness and validation of our analytic method in the analysis of general models. A practical application of the method is also proposed to the uncertainty and sensitivity analysis of a deterministic HIV model.

1. Introduction

Mathematical models are of great importance in the natural sciences, especially in physics. They help one explicate a social system in mathematical language, analyze the roles of linked parameters by physical methods, and then make predictions about underlying behaviors. In mathematical modeling, uncertainty surrounding the model response is usually qualitatively handled by changing one parameter at a time in keeping the remaining variables constant [1–6]. This widely used analysis method is performed by treating input parameters as random variables and by neglecting the interaction effects between

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different parameters. With the introducing of global uncertainty and sensitivity analysis, the roles of individual parameters are then quantified in the evaluation of model response.

Currently the global uncertainty and sensitivity analysis is widely performed in various disciplines involving social science [7,8], engineering science [9], economics [10], chemistry [11], physics [12–15], etc. It has been stated as a formal tool for statistical evaluation of models and often contributes to models' quality and application. Specifically, the global uncertainty and sensitivity analysis is beneficial for gaining insight into how input parameters can be ranked according to their importance in establishing the uncertainty of model response.

At present, many strategies have been built for the implementation of uncertainty and sensitivity analysis, including the above mentioned changing one factor at a time [16,17], local method [18–21], regression analysis [22], variance-based method [23], etc. Among the various available strategies, variance-based sensitivity analysis has been assessed as versatile and effective for uncertainty and sensitivity analysis of complex models. The consideration of variance-based importance measures can be traced back to over twenty years ago when Sobol characterized the first-order sensitivity measures on the basis of deposing the variance in model response into different partial contributions attributable to individual input variables and to their combinations (called variance decomposition) [24]. Then extensive relevant investigations are carried out around this Sobol's work, boiling down to the improvements in analysis strategies and to their applications to the sensitivity and reliability analysis of complex systems [25,26]. However, these frameworks are often performed when the input variables are assumed to be statistically independent of each other.

Recently, the interest in extending sensitivity analysis strategies from uncorrelated case to the correlated one is increasing due to the existence of correlated input factors in many practical applications. Previous investigations on sensitivity analysis of models with correlated input variables only provided overall sensitivity indices with respect to individual factors. However, considerations of the correlated and independent variance contributions were absent [27]. In practical applications, the distinction between independent and correlated variance contributions is quite important. It allows one to decide whether or not the correlations among input factors should be considered.

Both the correlated and independent variance contributions were firstly considered by Xu et al. [28]. They proposed a regression-based strategy to decompose partial variance contributions into independent and correlated parts assuming linear relationship connecting the model response and input variables. To overcome the limitation of their method, many frameworks on sensitivity analysis are recently developed in the presence of correlated parameters, contraposing the investigation of more effective and universal technics for sensitivity analysis in general correlated situations [29–31]. Still, a theoretical framework for the determination of partial variance contributions and of relative effects contributed by the independence and correlations of input variables is limited, especially when a single input is correlated with many others simultaneously.

In this work, an analytic method is built for the uncertainty and sensitivity analysis of models with correlated input parameters. A mathematical expression is firstly derived to characterize the variance propagation from correlated parameters to the model response. The expression provides fruitful information for evaluating partial variance contributions produced by individual variables alone and also by their interactions to the model response. By using linear correlation model, a single variable can be divided into independent and correlated sections. Universal expressions are then proposed for the coefficients that specify the independent and correlated sections of an arbitrary variable. The coefficients, together with the expressions of variance propagation, serve to identify the sensitivity of model response with respect to the independence and correlations of input parameters. Furthermore, except for the independent and correlated variance contributions, the partial variance produced by the coupling effect between input independence and correlations is also identified from the total variance of model response.

The rest of the paper is organized as follows. In Section 2, the analytic formula for variance propagation is generalized to the case of correlated input variables. The independent, correlated and coupling variance contributions are also interpreted in this section. Sensitivity measures are defined in Section 3, in the presence of input correlations. The generation process of correlated variables is also analyzed here, by using linear correlation model. In Section 4, four fabricated numerical models illustrate the effectiveness and applicability of our analytic framework, accompanied by a practical application to the sensitivity analysis of a deterministic HIV model. Section 5 gives concluding remarks.

2. Variance propagation

Any operation that we perform on a model response dependent upon variables of uncertainty requires us to identify the response uncertainty based on the uncertainty in input variables. The propagation of variance, characterizing the effect of input uncertainty on the uncertainty of model response, constitutes the essential ingredient of uncertainty and sensitivity analysis of complex models.

2.1. Independent case

Consider a general mathematical model of the form $y = f(\boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ labeling the input vector of n-dimensional variables of uncertainty. The Taylor series of model response y at the center point of input vector is