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On the existence of accessibility in a tree-indexed percolation model

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HIGHLIGHTS

- We establish new properties of the accessibility percolation model on infinite trees.
- We suggest a martingale approach to study accessibility percolation on tree-like graphs.
- We state a connection between accessibility percolation and records.

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ABSTRACT

We study the accessibility percolation model on infinite trees. The model is defined by associating an absolute continuous random variable X_v to each vertex v of the tree. The main question to be considered is the existence or not of an infinite path of nearest neighbors $v_1, v_2, v_3 \ldots$ such that $X_{v_1} < X_{v_2} < X_{v_3} < \cdots$ and which spans the entire graph. The event defined by the existence of such path is called *percolation*.

We consider the case of the accessibility percolation model on a spherically symmetric tree with growth function given by $f(i) = \lceil (i+1)^{\alpha} \rceil$, where $\alpha > 0$ is a given constant. We show that there is a percolation threshold at $\alpha_c = 1$ such that there is percolation if $\alpha > 1$ and there is absence of percolation if $\alpha \leq 1$. Moreover, we study the event of percolation starting at any vertex, as well as the continuity of the percolation probability function. Finally, we provide a comparison between this model with the well known F^{α} record model. We also discuss a number of open problems concerning the accessibility percolation model for further consideration in future research.

1. Introduction

Many percolation models were inspired by physics and developed in order to answer questions of physical and mathematical interest. This summarizes, roughly speaking, the beginning of a modern theory with interesting rigorous results and a wide range of applications. The standard model of percolation appears as a mathematical model for the first time in the work of Broadbent and Hammersley in 1957 — see [1]. The main purpose of that work was to study how random properties of a porous medium influence the transport of a fluid through it. In order to accomplish it, a lattice is used to

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represent the medium, where vertices are associated to pores and bonds to channels, and a family of Bernoulli independent and identically distributed random variables with parameter p is used to represent when a given channel is open or not to the spread of a fluid. The first question to be considered is the existence or not of an infinite component of open channels and how this event depends on p. The answer gives rise to a result known as phase transition behavior, which guarantees that there exists a nontrivial critical value of the parameter p called p_c such that if p is above p_c the origin belongs to an infinite connected component with positive probability. The reader can find more details about mathematical formulation and main results for the standard percolation model and related spatial processes in [2–4].

The resulting mathematical theory was quickly developed and became one of the main branches of contemporary probability. On the other hand, new percolation models appear in the literature as an alternative to understand issues of biological and physical interest. Some instances are the *AB* percolation, invasion percolation, oriented percolation, and continuum percolation models, among many others — for more references see [3,5]. As a theoretical tool, combined with coupling techniques, percolation theory is also useful in the construction and analysis of stochastic processes [6–9].

The accessibility percolation model in trees was introduced in [10] inspired by questions from evolutionary biology. In that work an n-tree is considered and a continuous random variable X_v is associated to each vertex v, independently of everything else. One of the main questions in this model is whether there exists a path of nearest neighbors $v_1, v_2, v_3 \dots$ such that

$$X_{v_1} < X_{v_2} < X_{v_3} < \cdots$$

with positive probability. This type of path is called *accessible path*. In [10] the authors derived an asymptotic result for the probability of having at least one accessible path connecting the root with the kth-level of an $n(k)$-tree, with $n(k) := \alpha\, k$, and α some arbitrary positive constant. Thus, they proved the existence of a percolation threshold for this model as $k \to \infty$. Indeed, they showed that the probability of having at least one accessible path goes to zero for $\alpha < \alpha_c$ and converges to a positive number for $\alpha > \alpha_c$ with $1/e \le \alpha_c \le 1$. Later, this result was complemented in [11] where the authors showed that this probability converges to 1 for $\alpha > 1/e$. Recently, a related problem was analyzed in the hypercube in [12,13].

In this work we establish some properties of the accessibility percolation model on spherically symmetric trees. It is a well known fact that the study of tree-indexed stochastic processes is of interest in understanding evolutionary biological questions. An instance of such a process is obtained by considering a phylogenetic or evolutionary tree, with vertices representing species and edges representing evolutionary relationships among them, and associating a (random) fitness value to each individual. Our model may be seen as a simple stochastic model for evolutionary trees. Different levels of the spherically symmetric tree represent different *generations* of species and the degree of each vertex represents the number of *offspring* species which appear in fixed intervals of time. In this sense, the varying environment of the tree is translated as a varying mutation rate. Related stochastic models for phylogenetic trees are proposed in [14,15]. In [15] the authors propose a model which considers a birth and death component and a fitness component. Then, depending on the value of the (constant) mutation rate, they obtain a phylogenetic tree consistent with an influenza tree and also with an HIV tree.

The paper is organized as follows. Section 2 gathers the formal notations and definitions of the model and states some basic results. A discussion about the phase transition of the model is included in Section 3 and Section 4 is devoted to a connection with the theory of records. Last section proposes a martingale approach to deal with this type of models. Indeed, such approach constitutes an alternative to the methods previously used in the literature to analyze the accessibility percolation model.

2. The model and basic results

2.1. Trees

We consider an infinite, locally finite, rooted tree $T = (\mathcal{V}, \mathcal{E})$. We denote the root of T by $\mathbf{0}$. Here $\mathcal{V}$ stands for the set of vertices and $\mathcal{E} \subset \{\{u, v\} : u, v \in \mathcal{V}, u \neq v\}$ stands for the set of edges. If $\{u, v\} \in \mathcal{E}$, we say that u and v are neighbors, which is denoted by $u \sim v$. The degree of a vertex v, denoted by $d(v)$, is the number of its neighbors. A path in T is a finite sequence $v_0, v_1, \dots, v_n$ of distinct vertices such that $v_i \sim v_{i+1}$ for each i. Since T is a tree, there is a unique path connecting any pair of distinct vertices u and v. Therefore we may define the distance between them, which is denoted by $d(u, v)$, as the number of edges in such path. For each $v \in \mathcal{V}$ define $|v| := d(\mathbf{0}, v)$.

For $u, v \in \mathcal{V}$, we say that $u \le v$ if u is one of the vertices of the path connecting $\mathbf{0}$ and v; $u < v$ if $u \le v$ and $u \neq v$. We call v a *descendant* of u if $u \le v$ and denote by $T^u = \{v \in \mathcal{V} : u \le v\}$ the set of descendants of u. On the other hand, v is said to be a *successor* of u if $u \le v$ and $u \sim v$. For $n \ge 1$, we denote by ∂T_n the set of vertices at distance n from the root. That is, $\partial T_n = \{v \in \mathcal{V} : |v| = n\}$.

In this work we deal with spherically symmetric trees (SST) which are trees where the degree of any vertex depends only on its distance from the root. In other words, $d(\mathbf{0}) = f(0)$ for any $v \in \mathcal{V}$ we have $d(v) = f(|v|) + 1$ where $f := (f(i))_{i\ge 0}$ is a given sequence of natural numbers. The function f is called the growth function of the tree. Such trees will be denoted by T_f. We point out that there is no more information in T_f than that contained in the sequence $(|\partial T_n|)_{n\ge 1}$.