Systems & Control Letters 95 (2016) 3–10

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Reprint of "The Kalman–Yakubovich–Popov inequality for differential-algebraic systems: Existence of nonpositive solutions"[☆]

Timo Reis[a,*], Matthias Voigt[b]

[a] Universität Hamburg, Fachbereich Mathematik, Bundestraße 55, 20146 Hamburg, Germany
[b] Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Available online 2 June 2016

Keywords:
Differential-algebraic equation
Kalman–Yakubovich–Popov lemma
Popov function
Bounded real lemma
Positive real lemma

ABSTRACT

The Kalman–Yakubovich–Popov lemma is a central result in systems and control theory which relates the positive semidefiniteness of a Popov function on the imaginary axis to the solvability of a linear matrix inequality. In this paper we prove sufficient conditions for the existence of a nonpositive solution of this inequality for differential-algebraic systems. Our conditions are given in terms of positivity of a modified Popov function in the right complex half-plane. Our results also apply to non-controllable systems. Consequences of our results are bounded real and positive real lemmas for differential-algebraic systems.

1. Introduction

We consider linear time-invariant differential-algebraic control systems

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1}$$

where $sE - A \in \mathbb{K}[s]^{n\times n}$ is assumed to be *regular* (i. e., $\det(sE - A)$ is not the zero polynomial) and $B \in \mathbb{K}^{n\times m}$. The set of such systems is denoted by $\Sigma_{n,m}(\mathbb{K})$ and we write $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. The function $u : \mathbb{R} \to \mathbb{K}^m$ is a *control input*, whereas $x : \mathbb{R} \to \mathbb{K}^n$ denotes the state of the system. The set of all solution trajectories $(x, u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ induces the *behavior*

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m) : \frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu \right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the distributional derivative.

We consider the so-called *modified Popov function*

$$\Psi : \mathbb{C} \setminus \sigma(E, A) \to \mathbb{C}^{m\times m}, \tag{2a}$$

where $\sigma(E, A) = \{\lambda \in \mathbb{C} : \det(\lambda E - A) \neq 0\}$ (see the subsequent section for the notation) and

$$\Psi(\lambda) = \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \tag{2b}$$

with $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$. Note that on $\mathrm{i}\mathbb{R}$, $\Psi(\cdot)$ attains Hermitian values and coincides with the classical Popov function (where the λ in the first factor is replaced by $-\overline{\lambda}$). In contrast to the Popov function, $\Psi(\cdot)$ is neither rational nor meromorphic.

First we revisit a characterization for $\Psi(\cdot) \geq 0$ on the imaginary axis which is strongly related to the feasibility of infinite time horizon linear–quadratic optimal control problems with *zero final state* [1]. For standard state space systems (i. e., with $E = I_n$), the above property can be checked by the famous Kalman–Yakubovich–Popov (KYP) lemma, see [2–5]. The lemma states that if $[I_n, A, B]$ is controllable, then $\Psi(\mathrm{i}\omega) \geq 0$ holds true for all $\mathrm{i}\omega \notin \sigma(A)$ if and only if the so-called *KYP inequality*

$$\begin{bmatrix} A^*P + PA + Q & PB + S \\ B^*P + S^* & R \end{bmatrix} \geq 0 \tag{3}$$

has a Hermitian solution $P \in \mathbb{K}^{n\times n}$.

On the other hand, there are modifications of this lemma for special choices of Q, S, and R. For instance, for $Q = 0_{n\times n}$, $S = C^*$, and $R = D + D^*$, one can show that if $[I_n, A, B]$ is controllable, then with $G(s) = C(sI_n - A)^{-1}B + D$ it holds that

$$\Psi(\lambda) = G(\lambda) + G(\lambda)^* \geq 0 \quad \forall \lambda \in \mathbb{C}^+ \setminus \sigma(A)$$

if and only if the KYP inequality (3) with $Q = 0_{n\times n}$, $S = C^*$, and $R = D + D^*$ has a solution $P \leq 0$. This result is called *positive real lemma* and is of great importance in the context of passivity [6].

DOI of original article: http://dx.doi.org/10.1016/j.sysconle.2015.09.003.

☆ This article is a reprint of a previously published article. For citation purposes, please use the original publication details [Systems and Control Letters, 86 (2015), pp. 1–8].

* Corresponding author.
E-mail addresses: timo.reis@math.uni-hamburg.de (T. Reis), mvoigt@math.tu-berlin.de (M. Voigt).

http://dx.doi.org/10.1016/j.sysconle.2016.05.010

The KYP lemma states that positive semi-definiteness of $\Psi(\cdot)$ on $\mathrm{i}\mathbb{R} \setminus \sigma(A)$ is equivalent to the existence of *a* solution of the KYP inequality, whereas for the positive real lemma, positive semi-definiteness of $\Psi(\cdot)$ in $\mathbb{C}^+ \setminus \sigma(A)$ is equivalent to the existence of a *nonpositive* solution of the KYP inequality.

Thus, a natural question is whether $\Psi(\cdot) \geq 0$ in $\mathbb{C}^+ \setminus \sigma(A)$ is equivalent to the existence of a solution $P \leq 0$ of the KYP inequality. The great Jan C. Willems first casually claimed in his seminal article [1] that this statement holds true for controllable systems. However, in a successive erratum [7], this claim has been disproved by himself with the aid of a counter-example. Willems further stated in this erratum [7] that the equivalence holds, if an inertial decomposition

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \begin{bmatrix} C_1^*C_1 & C_1^*D_1 \\ D_1^*C_1 & D_1^*D_1 \end{bmatrix} - \begin{bmatrix} C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix}, \tag{4a}$$

exists, where $C_1 \in \mathbb{K}^{m\times n}$, $C_2 \in \mathbb{K}^{p_2\times n}$, $D_1 \in \mathbb{K}^{m\times m}$, $D_2 \in \mathbb{K}^{p_2\times m}$, and

$$G_1(s) := C_1(sI_n - A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)). \tag{4b}$$

However, no proof of this statement has been carried out. In [8], it has been proven that for controllable and stable systems with the additional property that the inverse of $G_1(s)$ is bounded in $\mathbb{C}_+$, all solutions of the KYP inequality are positive definite.

A further condition for the larger class of behavioral systems has been examined by Trentelman and Rapisarda in [9,10]. Under an additional assumption which translates to $\Psi(\cdot) > 0$ on $\mathrm{i}\mathbb{R}$, the existence of nonpositive solutions has been characterized by means of an associated *Pick matrix*. In this paper, we revisit Willems' condition (4) and prove it for differential-algebraic systems. Thereby we are also dealing with non-controllable systems. We further present conditions for all solutions of the KYP inequality being nonnegative. We will apply our results to formulate positive real and bounded real lemmas for differential-algebraic systems.

Notation

We use the standard notations i, $\overline{\lambda}$, A^*, A^{-*}, I_n, $0_{m\times n}$ for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix and its inverse, the identity matrix of size $n \times n$ and the zero matrix of size $m \times n$ (subscripts may be omitted, if clear from context). The symbol $\mathbb{K}$ stands for either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. The closure of $S \subset \mathbb{C}$ is denoted by $\overline{S}$.

By writing $A \geq (\leq) B$ we mean that for two Hermitian matrices $A, B \in \mathbb{K}^{n\times n}$, the matrix $A - B$ is positive semidefinite (negative semidefinite). The following concept, namely equality and semidefiniteness on some subspace will be frequently used in this article.

Definition 1.1 (*Equality and Semidefiniteness on a Subspace*)**.** Let $\mathcal{V} \subseteq \mathbb{K}^n$ be a subspace and $A, B \in \mathbb{K}^{n\times n}$ be Hermitian. Then we write

$$A =_{\mathcal{V}} (\geq_{\mathcal{V}}, \leq_{\mathcal{V}})\, B,$$

if we have $v^*(A - B)v = (\geq, \leq)\, 0$ for all $v \in \mathcal{V}$.

The following sets are further used in this article:

$\mathbb{N}_0$	the set of natural numbers including zero
$\mathbb{C}^+, \mathbb{C}^-$	the open sets of complex numbers with positive and negative real parts, resp.
$\mathbb{K}[s], \mathbb{K}(s)$	the ring of polynomials and the field of rational functions with coefficients in $\mathbb{K}$, resp.
$\mathrm{Gl}_n(\mathcal{K})$	the group of invertible $n \times n$ matrices with entries in a field $\mathcal{K}$
$\sigma(A)$	spectrum of $A \in \mathbb{K}^{n\times n}$
$\sigma(E, A)$	$= \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\}$, the set of generalized eigenvalues of the matrix pencil $sE - A \in \mathbb{K}[s]^{n\times n}$
$\mathcal{RH}_\infty^{p\times m}$	the space of rational $p \times m$ matrix-valued functions which are bounded in $\mathbb{C}^+$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{K}^n)$	the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathbb{K}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$.

2. Preliminaries

2.1. Differential-algebraic systems

We first introduce some systems theoretic concepts for differential-algebraic systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. First we consider notions related to controllability and stabilizability, see also [11,12] and [13, Def. 5.2.2] for the definition and the respective algebraic conditions in terms of the system matrices.

Definition 2.1 (*Controllability and Stabilizability*)**.** A system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is called

(a) *behaviorally (beh.) stabilizable* if for all $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$, there exists some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with

$$(x(t), u(t)) = (x_1(t), u_1(t)) \quad \text{if } t < 0 \quad \text{and}$$
$$\lim_{t\to\infty} (x(t), u(t)) = 0;$$

(b) *behaviorally (beh.) controllable* if for all $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$, there exist some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ and some $T > 0$ with

$$(x(t), u(t)) = \begin{cases} (x_1(t), u_1(t)), & \text{if } t < 0, \\ (x_2(t), u_2(t)), & \text{if } t > T; \end{cases}$$

(c) *completely controllable* if for all $x_0, x_{\mathrm{f}} \in \mathbb{K}^n$, there exist some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ and some $T > 0$ with $x(0) = x_0$ and $x(T) = x_{\mathrm{f}}$.

Moreover, we also consider differential-algebraic systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ which have an additional *output equation*

$$y(t) = Cx(t) + Du(t),$$

where $C \in \mathbb{K}^{p\times n}$ and $D \in \mathbb{K}^{p\times m}$. We denote the set of all such systems by $\Sigma_{n,m,p}(\mathbb{K})$ and we write $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ (or $[E, A, B, C] \in \Sigma_{n,m,p}(\mathbb{K})$ if $D = 0$). The behavior is given by

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{(x, u, y) \in \mathfrak{B}_{[E,A,B]} \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^p) : y = Cx + Du\right\},$$

and the expression

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$$

is called the *transfer function of* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$.

Behavioral detectability means that the state can be asymptotically reconstructed from the knowledge of input and output, cf. [13, Def. 5.3.16]. See also [13, Thm. 5.3.17] for an equivalent algebraic criterion.

Definition 2.2 (*Behavioral Detectability*)**.** The system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is called *behaviorally (beh.) detectable* if

$$(x_1, u, y),\ (x_2, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \ \Rightarrow\ \lim_{t\to\infty} (x_1(t) - x_2(t)) = 0.$$

2.2. System equivalence form, system space, and space of consistent initial differential variables

In this paper we consider systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ under system equivalence which is defined as follows.