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Behavioral controllability and coprimeness for pseudorational transfer functions

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ABSTRACT

Controllability plays various crucial roles in behavioral system theory. While there exist several characterizations of this notion, in terms of the Bézout identity, image representation, direct sum decomposition, etc., its overall picture for infinite-dimensional systems still remains rather incomplete, in spite of various existing attempts. This article gives an extension of such results in a well-behaved class of infinite-dimensional systems, called pseudorational. A proper choice of an algebra makes the treatment more transparent. We establish equivalent conditions for controllability in terms of the Bézout identity, relationships with notions such as image representation and direct sum decompositions.

1. Introduction

Since this is a special issue dedicated to our friend late Jan Willems, I would like to start by explaining some background of the present work. In the conference paper [1], Jan and I published a work on controllability of infinite-dimensional behaviors defined over a ring of distributions. Roughly speaking, the work gave several equivalent conditions for behavioral controllability, albeit for scalar systems, that is, for behaviors defined by 1×2 kernel matrices. Although in such a limited context, the obtained results exhibit interesting relationships among controllability, coprimeness, image representations, direct sum decompositions.

Unfortunately, the proof contained some errors, and requires a further elaborate study. These errors were pointed out by Oberst [2], and I am grateful for it. Further unfortunately, we could not find a suitable correction while Jan was alive, but a proper fix was obtained only after his passing away. I would like to dedicate this short note to his memory and the long-term friendship and collaboration.[1] I hereby dedicate this work to Jan Willems.

This paper addresses various characterizations of behavioral controllability for infinite-dimensional systems, particularly in the context of the class called *pseudorational*. While there are a number of important contributions on controllability for infinite-dimensional systems, particularly in the context of delay-differential systems, e.g., [3–7,1], the overall picture in this context still remains open. We address this issue for behaviors defined in the class of pseudorational transfer functions [8–10,1]. A central idea is to introduce behaviors defined over a ring of distributions, and consider controllability and related properties there. Our main result is Theorem 3.4, where we establish various equivalent characterizations for behaviors defined over distributions. We give a discussion on the relationship with some existing results in Section 5.

Notation and Convention

$\mathscr{D}'$ is the space of distributions (in the sense of Schwartz) on $(-\infty, \infty)$. $\mathscr{D}'_+(\mathbb{R})$ denotes its subspaces having support bounded on the left; similarly, $\mathscr{D}'_-(\mathbb{R})$ is the subspace having support bounded on the right. $\mathscr{E}'(\mathbb{R})$ denotes the subspace of $\mathscr{D}'_+(\mathbb{R})$ (and also of $\mathscr{D}'_-(\mathbb{R})$ and $\mathscr{D}'$) with *compact* support. $\mathscr{E}'(\mathbb{R}_-)$ is the subspace of $\mathscr{E}'(\mathbb{R})$ consisting of those with support contained in the negative half line $(-\infty, 0]$. Each of these spaces constitutes a convolution algebra. These notions are standard, and the reader is referred to Schwartz [11,12]. $L^2_{loc}(-\infty, \infty)$ denotes the space of locally square integrable (with respect to the Lebesgue measure) functions. Distributions such as Dirac's delta δ_a placed at $a < 0$, its derivative δ'_a are examples of elements in $\mathscr{E}'(\mathbb{R}_-)$.

The *shift operator* in $\mathscr{D}'$ is defined by

$$(\sigma_t w)(s) := \delta_{-t} * w, \quad t \in \mathbb{R}. \tag{1}$$

If $t > 0$, it is the *left shift* and if $t < 0$ it is the *right shift* operator.

A distribution α is said to be of *order at most* m if it can be extended as a continuous linear functional on the space of m-times

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[1] Hence, naturally, this paper is based on [1] and has some overlaps with the developments, but the paper gives a revised proof of our main result.

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continuously differentiable functions. Such a distribution is said to be of *finite order*. The largest number m, if one exists, is called the *order* of α, and denoted by ord α [11,12]. The delta distribution δ_a, $a \in \mathbb{R}$ is of order zero, while its derivative δ_a' is of order one, etc. A distribution with compact support is known to be always of finite order [11,12].

For a distribution $\alpha \in \mathscr{D}'_+(\mathbb{R})$, define a real number $\ell(\alpha)$ by

$$\ell(\alpha) := \inf\{t \mid t \in \operatorname{supp}\alpha\}. \tag{2}$$

Similarly, for $\beta \in \mathscr{D}'_-(\mathbb{R})$,

$$r(\beta) := \sup\{t \mid t \in \operatorname{supp}\beta\}. \tag{3}$$

It follows from Titchmarsh's convolution theorem (local version) [13, p. 224] that

$$\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta) \tag{4}$$

$$r(\chi * \psi) = r(\chi) + r(\psi) \tag{5}$$

for $\alpha, \beta \in \mathscr{D}'_+(\mathbb{R})$ and $\chi, \psi \in \mathscr{D}'_-(\mathbb{R})$.

For a distribution $f \in \mathscr{D}'$, its Laplace transform $\hat{f}$ is defined by

$$\hat{f}(s) := \langle f, e^{-st}\rangle_t. \tag{6}$$

Here $\langle f, e^{-st}\rangle_t$ denotes the action of $f \in \mathscr{D}'$ on e^{-st} in the variable t, if it exists. Every $f \in \mathscr{E}'(\mathbb{R})$ has the Laplace transform. Note, in particular, $(\delta' - \lambda\delta)\hat{} = s - \lambda$.

2. Pseudorationality

The following definition is fundamental to our subsequent developments [8]:

Definition 2.1. Let R be a $\mathtt{p} \times \mathtt{w}$ matrix ($\mathtt{w} \geq \mathtt{p}$) with entries in $\mathscr{E}'(\mathbb{R})$. It is said to be *pseudorational* if there exists a $\mathtt{p} \times \mathtt{p}$ submatrix P such that

1. $P^{-1} \in \mathscr{D}'_+(\mathbb{R})$ exists with respect to convolution;
2. $\operatorname{ord}(\det P^{-1}) = -\operatorname{ord}(\det P)$.

We now give the following definition:

Definition 2.2. Let R be pseudorational as defined above. The *behavior* $\mathscr{B}$ defined by R is given by

$$\mathscr{B} := \{w \in (L^2_{loc}(-\infty,\infty))^{\mathtt{w}} \mid R * w = 0\}. \tag{7}$$

The *distributional behavior* $\mathscr{B}_{\mathscr{D}'}$ defined by R is given by

$$\mathscr{B}_{\mathscr{D}'} := \{w \in (\mathscr{D}')^{\mathtt{w}} \mid R * w = 0\}. \tag{8}$$

If we need to show the dependence on R explicitly, we will write $\mathscr{B}[R]$ or $\mathscr{B}_{\mathscr{D}'}[R]$. The convolution $R * w$ is of course taken in the sense of distributions. Since R has compact support, this convolution is always well defined [11].

The behavior $\mathscr{B}$ or $\mathscr{B}_{\mathscr{D}'}$ is *time-invariant* or *shift-invariant* in the sense that $\sigma_t\mathscr{B} \subset \mathscr{B}$ and $\sigma_t\mathscr{B}_{\mathscr{D}'} \subset \mathscr{B}_{\mathscr{D}'}$ for every $t \in \mathbb{R}$, where σ_t is the shift operator defined by (1). This follows clearly from the definition (7) since $R * (\sigma_t w) = R * \delta_{-t} * w = \delta_{-t} * R * w = 0$.

Pseudorationality has the advantage [8] that it provides a convenient state-space realization procedure. See also [10] for a concise survey.

3. Controllability and coprimeness

We start with the notion of controllability [14,22] in the present context.

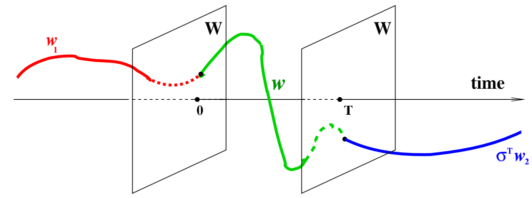

Fig. 1. Concatenation of trajectories.

Definition 3.1. Let R be pseudorational, and $\mathscr{B}$ the behavior associated to it. $\mathscr{B}$ is said to be *controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t - T)$ for $t \geq T$ (see Fig. 1). Let $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (8). $\mathscr{B}_{\mathscr{D}'}$ is said to be *distributionally controllable* if for every pair $w_1, w_2 \in \mathscr{B}_{\mathscr{D}'}$, there exists $T \geq 0$ and $w \in \mathscr{B}_{\mathscr{D}'}$, such that $w|_{(-\infty,0)} = w_1$ on $(-\infty, 0)$, and $w|_{(T,\infty)} = \sigma_{-T}w_2$ on (T, ∞).

In other words, every pair of trajectories can be concatenated into one trajectory that agrees with their respective past and future trajectories.

We now introduce various notions of coprimeness.

Definition 3.2. The pair (P, Q), $P \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{p}\times\mathtt{p}}$, $Q \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{p}\times\mathtt{m}}$ is said to be *spectrally coprime* if $\hat{P}(s)$ and $\hat{Q}(s)$ have no common zeros, i.e.,

$$\operatorname{rank}\begin{bmatrix}\hat{P}(\lambda) & \hat{Q}(\lambda)\end{bmatrix} = p, \quad \forall\lambda \in \mathbb{C}.$$

It is *approximately coprime* if there exist sequences $\Phi_n \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{p}\times\mathtt{p}}$, $\Psi_n \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{m}\times\mathtt{p}}$ such that $P * \Phi_n + Q * \Psi_n \to \delta I$ in $(\mathscr{E}'(\mathbb{R}))^{\mathtt{p}\times\mathtt{p}}$. The pair (P, Q) is said to satisfy the *Bézout identity* (or simply *Bézout*), if there exist $\Phi \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{p}\times\mathtt{p}}$ and $\Psi \in (\mathscr{E}'(\mathbb{R}))^{\mathtt{m}\times\mathtt{p}}$ such that

$$P * \Phi + Q * \Psi = \delta I. \tag{9}$$

Or equivalently,

$$\hat{P}(s)\hat{\Phi}(s) + \hat{Q}(s)\hat{\Psi}(s) = I \tag{10}$$

for some entire functions $\hat{\Phi}, \hat{\Psi}$ satisfying the Paley–Wiener estimate (A.1).

Remark 3.3. The above definitions carry over to the special case when we confine ourselves to the subalgebra $\mathscr{E}'(\mathbb{R}_-)$. This case corresponds to the various state space properties of causal state space representations [9]. For example, spectral coprimeness is equivalent to controllability (reachability) of every eigen-mode of a realization, and approximate coprimeness to approximate reachability of the realization [9]. In this case, there is a gap between the two, since there can be an unreachable subspace that is not spanned by eigenspaces. For example, if $P, Q \in \mathscr{E}'(\mathbb{R}_-)$ are such that $P = \delta_a P_0$ and $Q = \delta_a Q_0$ with $P_0, Q_0 \in \mathscr{E}'(\mathbb{R}_-)$, $a < 0$, then (P, Q) is *not* approximately coprime over $\mathscr{E}'(\mathbb{R}_-)$ since it admits an redundant delay element corresponding to δ_a, even if it may be spectrally coprime. In this case, one may have [9]

$$P * \Phi_n + Q * \Psi_n \to \delta_a I, \quad a > 0. \tag{11}$$

Such a δ_a corresponds to a pure delay in the state space that is to be canceled by time-delay in the output. While this is crucial in the state space representation, it makes no difference in the behavior. In fact, since δ_a is invertible in $\mathscr{E}'(\mathbb{R})$, (11) implies approximate coprimeness. In fact, this is the only case where the gap between spectral coprimeness and approximate coprimeness occurs [9], hence one can conclude the two coprimeness notions coincide for $\mathscr{E}'(\mathbb{R})$. See also [15] for pertinent results.