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Lossless H_∞-synthesis for 2D systems (special issue JCW)

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ABSTRACT

We study 2D systems with a continuous and discrete time axis. We embed known results about the stability and H_∞-performance properties of such systems into multiplier theory from robust control. It is shown that this opens the way for applying recently developed gain-scheduled controller synthesis techniques in order to solve the H_∞-design problem for 2D systems without any conservatism. This is presented if the discrete-time axis is either one- or two-sided, the latter leading to non-conservative H_∞-synthesis result for infinite string interconnections of identical linear time-invariant systems.

1. Introduction

Multidimensional or nD systems theory is a rather well-developed field, with applications to e.g. two-dimensional filtering in image processing, to analyzing and synthesizing controllers for repetitive processes or for spatially distributed systems, to name only a few. The last decades have witnessed a whole variety of techniques for analyzing the stability and performance properties of 2D or nD system, either by algebraic, functional analytic or optimization techniques. A multitude of classical papers address the characterization of stability as surveyed e.g. in [1] and devoted to 2D systems admitting a Roesser- [2,3] or Fornasini–Marchesini state-space description [4]. Already in this early work the prominent role of the existence of positive definite frequency-dependent solutions of suitable parameterized Lyapunov equations for exact stability characterizations has become apparent. The interpretation in terms of quadratic differential forms in a behavioral setting has been made transparent by Jan Willems in [5], see also [6–9].

In a similar vein, the solution of the infinite horizon linear quadratic control problem involves the solution of parameterized Riccati equations as exposed e.g. in [10]. A lossless characterization of some bound on the H_∞-norm of a 2D system can as well be achieved through frequency-parameterized Riccati equations that are related to the bounded real lemma, as brought out in the context of spatially invariant distributed systems in [11,12]. A related computational technique based on parameterized linear matrix inequalities (LMIs) has been recently developed in [13,14], see also [15].

In general, the special role of LMIs for the computational analysis of multidimensional systems and for designing multidimensional filters or controllers is by now well recognized, as e.g. emphasized for linear repetitive processes in [16]. Based on Lyapunov or dissipation theory, a large body of literature is devoted to a wide diversity of questions from filtering and control with uncertainties, non-linearities or delays. It is certainly not our intention to survey or summarize previous work in this direction but, instead, we rather mention the books [17,18] for a comprehensive overview and many more references.

Even if looking at H_∞-synthesis for 2D systems, all these techniques still involve conservatism since, roughly speaking, they are based on frequency-independent solutions of frequency dependent analysis inequalities.

This brings us to the main purpose of the present paper. In a first part, we emphasize the conceptual simplicity how tight stability and performance tests emerge by applying classical frequency-dependent multiplier techniques in robust control. This is pursued in detail for the particular class of 2D systems modeling linear repetitive processes with a continuous and a discrete time axis in a Roesser description in Section 2. In a next step we provide the link of lossless H_∞-synthesis for this class of systems to a recently developed gain-scheduling design algorithm with dynamic multipliers in [19,20] (Section 3). We conclude the paper with a novel extension of the design paradigm to two-sided spatially distributed systems (Section 4), accompanied by a discussion of variants and extensions (Section 5). Technical proofs are moved to Appendix.

Notation. We use the notation $\mathbb{C}_\geq := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$, $\mathbb{R}_\geq := \mathbb{C}_\geq \cap \mathbb{R}$, $\mathbb{Z}_\geq := \mathbb{R}_\geq \cap \mathbb{Z}$, $\mathbb{D}_\geq := \{z \in \mathbb{C} \mid |z| \geq 1\}$ and $\mathbb{C}_\geq^\infty := \mathbb{C}_\geq \cup \{\infty\}$, $\mathbb{D}_\geq^\infty := \mathbb{D}_\geq \cup \{\infty\}$ as well as the analogous versions for "$>$", "$<$" and "$\leq$", respectively. All throughout $\|\cdot\|$ denotes the

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Euclidean norm of vectors or the spectral norm of matrices. Further $RL_\infty^{k\times l}$ ($RL_\infty^{k\times l}(\mathbb{D}_=)$) is the space of real rational $k\times l$ matrices without poles in $\mathbb{C}_=^\infty$ ($\mathbb{D}_=$) and equipped with the maximum norm $\|.\|_\infty$. For a real rational G let $G^*(s) := G(-s)^T$. If $G(s) = D + C(sI - A)^{-1}B$, we write $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $G_{ss} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and use $(G^*)_{ss} = \left(\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right)$. If $M = M^T \in \mathbb{R}^{k\times k}$ and $G \in RL_\infty^{k\times l}$ is realized with $\mathrm{eig}(A)\cap\mathbb{C}_= = \emptyset$, the continuous-time Kalman–Yakubovich–Popov (KYP) Lemma states that the frequency domain inequality (FDI) $G^*MG \prec 0$ on $\mathbb{C}_=^\infty$ holds iff there exists some $X = X^T$ satisfying the linear matrix inequality (LMI)

$$\mathscr{L}(X, M, G_{ss}) := \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0. \quad (1)$$

Partitions of A induce compatible ones for X without notice. We use the star product "$\star$" and all rules for linear fractional transformations (LFTs) as in [21, Chapter 10]. Finally, "$\bullet$" indicates objects that can be inferred by symmetry or are irrelevant.

2. Analysis of 2D systems

2.1. Stability

With real matrices $A \in \mathbb{R}^{n\times n}$, $B, C, D \in \mathbb{R}^{m\times m}$ we consider the system

$$\begin{pmatrix} \partial_t x(t,k) \\ w(t,k+1) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t,k) \\ w(t,k) \end{pmatrix}$$
$$\text{for } (t,k) \in \mathbb{R}_\geq \times \mathbb{Z}_\geq. \quad (2)$$

A trajectory is a function $\mathrm{col}(x, w) : \mathbb{R}_\geq \times \mathbb{Z}_\geq \to \mathbb{R}^{n+m}$ which has a locally square integrable distributional partial derivative $\partial_t x$ with respect to the first variable and satisfies (2).

System (2) is said to be stable if

$$\det(sI - A)\det(zI - D) \neq 0 \quad \text{and}$$
$$\det\begin{pmatrix} A - sI & B \\ C & D - zI \end{pmatrix} \neq 0 \quad \text{for all } (s,z) \in \mathbb{C}_\geq \times \mathbb{D}_\geq. \quad (3)$$

This is the exact algebraic characterization for stability along the pass as defined in [16] for linear repetitive processes. The relation of (3) to exponential stability can be found in [22,13] (with a discussion of the deficiencies of this interpretation in [23], see also [24]).

By the Schur formula and with

$$H(s) := D + C(sI - A)^{-1}B \quad \text{as well as} \quad \delta(z) := \frac{1}{z},$$

the third determinant in (3) equals $\det(A - sI)\det(H(s) - zI)$, which can be expressed as $\det(A-sI)\det(\delta(z)I)\det(\delta(z)H(s)-I)$; hence (3) is equivalent to $\mathrm{eig}(A) \subset \mathbb{C}_<$, $\det(I - D\delta(z)) \neq 0$ and $\det(I - H(s)\delta(z)) \neq 0$ for all $(s,z) \in \mathbb{C}_\geq \times \mathbb{D}_\geq$; due to $H(\infty) = D$ and $\delta(\infty) = 0$, this is nothing but

$$\mathrm{eig}(A) \subset \mathbb{C}_< \quad \text{and}$$
$$\det(I - H(s)\delta(z)) \neq 0 \quad \text{for all } (s,z) \in \mathbb{C}_\geq^\infty \times \mathbb{D}_\geq^\infty. \quad (4)$$

This reformulation clearly exhibits the link of the stability condition (3) to the structured singular value (SSV) [25]. As well-known, one can confine the non-singularity condition in (4) to the boundary $\mathbb{C}_=^\infty$ of $\mathbb{C}_\geq^\infty$. (This follows either from a maximum modulus principle for the SSV [25] or by exploiting that $\delta(\mathbb{D}_\geq^\infty) = \mathbb{D}_\leq$ is star-shaped with center zero and using a homotopy argument in combination with the continuous dependence of the eigenvalues of a matrix on its entries.) Thus (4) is equivalent to $\mathrm{eig}(A) \subset \mathbb{C}_<$ and $\det(I - H(s)\delta(z)) \neq 0$ for all $(s,z) \in \mathbb{C}_=^\infty \times \mathbb{D}_\geq^\infty$, which can in turn be rephrased as

$$\mathrm{eig}(A) \subset \mathbb{C}_< \quad \text{and} \quad \mathrm{eig}(H(s)) \subset \mathbb{D}_< \quad \text{for all } s \in \mathbb{C}_=^\infty. \quad (5)$$

The second point-wise stability property in discrete time can alternatively be expressed by requiring, for each $s \in \mathbb{C}_=^\infty$, the existence of a positive definite solution Ψ of the Lyapunov inequality $H(s)^*\Psi H(s) - \Psi \prec 0$; note that Ψ clearly depends on s! Hence (5) is equivalent to $\mathrm{eig}(A) \subset \mathbb{C}_<$ and the existence of a function $\Psi : \mathbb{C}_=^\infty \to \mathbb{C}^{m\times m}$ satisfying the FDIs

$$\Psi(s) \succ 0 \quad \text{and}$$
$$\begin{pmatrix} H(s) \\ I \end{pmatrix}^* \begin{pmatrix} \Psi(s) & 0 \\ 0 & -\Psi(s) \end{pmatrix} \begin{pmatrix} H(s) \\ I \end{pmatrix} \prec 0 \quad \text{for all } s \in \mathbb{C}_=^\infty. \quad (6)$$

For such parameter-dependent linear matrix inequalities, it is finally well-established that it causes no loss of generality to let $\Psi(s)$ be real rational in s without poles in $\mathbb{C}_=^\infty$ [26,27]. We can summarize these observations as follows.

Lemma 1. *System (2) is stable iff $\mathrm{eig}(A) \subset \mathbb{C}_<$ and there exists a $\Psi \in RL_\infty^{m\times m}$ with (6).*

In this fashion stability is characterized by a convex feasibility test over the infinite-dimensional space $RL_\infty^{m\times m}$. Before we address the construction of an asymptotically exact finite-dimensional LMI relaxation hierarchy for verifying stability, let us discuss how our point-of-view seamlessly extends to performance characterizations.

Remark 2. We emphasize that none of the analysis results in this paper is new. For example, (5) appears in [28, Lemma 6]. If choosing the simple multiplier $\Psi(s) = I$ and applying the continuous-time KYP Lemma to turn (6) for $H(s)^T$ realized by (A^T, C^T, B^T, D^T) into an LMI, we arrive at [28, Theorem 2]. All this is closely linked to insights in many other papers in the literature, which are too numerous to be cited here.

2.2. Performance

To render the notation more compact, let σ denote the shift operator $\sigma w(t,k) = w(t,k+1)$ for $(t,k) \in \mathbb{R}_\geq \times \mathbb{Z}_\geq$ acting on the second variable of the signal $w : \mathbb{R}_\geq \times \mathbb{Z}_\geq \to \mathbb{R}^m$. For performance analysis we now consider the system

$$\begin{pmatrix} \partial_t x \\ \sigma w \end{pmatrix} = \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix} d,$$
$$e = \begin{pmatrix} C_2 & D_{21} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + D_{22} d \quad (7)$$

with a disturbance input d and an output e which is interpreted as an error variable. For $d = 0$ we obtain a system as in (2) whose stability is, in view of (5), characterized by

$$\mathrm{eig}(A) \subset \mathbb{C}_< \quad \text{and}$$
$$\mathrm{eig}(C_1(sI - A)^{-1}B_1 + D_{11}) \subset \mathbb{D}_< \quad \text{for all } s \in \mathbb{C}_=^\infty. \quad (8)$$

Let (8) be valid and let the disturbance be contained in the following Hilbert space of finite energy signals: $\mathscr{L}_2^\bullet = \{x : \mathbb{R}_\geq \times \mathbb{Z}_\geq \to \mathbb{R}^\bullet \mid \|x\|_2 := \sqrt{\sum_{k\in\mathbb{Z}_\geq} \int_{\mathbb{R}_\geq} \|x(t,k)\|^2\,dt} < \infty\}$; we often drop $\bullet$ for convenience. Then the output response of (7) with the initial conditions

$$x(0,k) = 0 \quad \text{and} \quad w(t,0) = 0 \quad \text{for all } k \in \mathbb{Z}_\geq,\ t \in \mathbb{R}_\geq$$