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Building systems from simple hyperbolic ones

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ABSTRACT

In this article we introduce a technique that derives from the existence and uniqueness of solutions to a simple hyperbolic partial differential equation (p.d.e.) the existence and uniqueness of solutions to hyperbolic and parabolic p.d.e.'s. Among others, we show that starting with an impedance passive system associated to the undamped wave equation, we can obtain an impedance passive system associated to the heat conduction equation.

1. Introduction

The class of state space models has become the standard class within control theory. It has been successfully applied in the study of finite- and infinite-dimensional linear systems. The latter will be the focus of this paper. Infinite-dimensional models describe systems given by, for instance, partial differential equations (p.d.e.'s) with control and observation within or at the boundary of their spatial domain. State space theory reformulates a p.d.e. as the abstract differential equation

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{1}$$

on a Hilbert or a Banach space, see e.g. [1,2]. Throughout this paper our state space will be a Hilbert space. Often Eq. (1) stems from physical modelling, in which A has the form:

$$A: \ x \mapsto \left(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*\right)(\mathcal{H}x), \tag{2}$$

where $\mathcal{J}$ is a formally skew-adjoint differential operator, $\mathcal{G}_R^*$ is the formal adjoint[1] of the differential operator $\mathcal{G}_R$, $S + S^*$ is a non-negative linear map and $\mathcal{H}$ is a positive linear map. This decomposition may be derived from a network type approach to physical system modelling called port Hamiltonian systems [3] where the map $\mathcal{H}$ corresponds to the energy density $x^\top \mathcal{H} x$ and the operator (2) is actually defined in a formally equivalent way by

$$\begin{pmatrix} x \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} z \\ z_2 \end{pmatrix} = \mathcal{J}_{\text{ext}} \begin{pmatrix} \mathcal{H} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ x_2 \end{pmatrix} \tag{3}$$

with the *interconnection structure* operator

$$\mathcal{J}_{\text{ext}} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \tag{4}$$

associated to the *closure relation*

$$x_2 = S z_2. \tag{5}$$

It is easily seen that (3)–(5) gives that the mapping form x to z is given by

$$\left(\mathcal{J} \ \mathcal{G}_R\right) \begin{pmatrix} I \\ -S \, \mathcal{G}_R^* \end{pmatrix} \mathcal{H} = \left(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*\right) \mathcal{H}, \tag{6}$$

which is the operator A of (2).

In this way we define a class of differential operators A which are parametrised by the operators $\mathcal{J}_{\text{ext}}$, S and $\mathcal{H}$. We illustrate this with a simple example.

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[1] The formal adjoint of the linear mapping Q is the linear mapping Q^* satisfying $\int_\Omega (Qf)^T g d\omega = \int_\Omega f^T (Q^* g)\, d\omega$ for all smooth functions f, g with compact support contained in the interior of the spatial domain Ω.

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Example 1.1. The dynamical model of the vibrating string with structural damping on the spatial interval $[a, b]$ is given by the following partial differential equation

$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial\zeta}\left[T(\zeta)\frac{\partial w}{\partial\zeta}(\zeta,t)\right] + \frac{\partial}{\partial\zeta}\left[k_s(\zeta)\frac{\partial^2 w}{\partial t\,\partial\zeta}(\zeta,t)\right], \tag{7}$$

where $\rho(\zeta)$ is the linear mass density, $T(\zeta)$ is the elasticity modulus and $k_s(\zeta)$ is the structural damping coefficient, all taking values in the interval $[m, M]$ with $0 < m \le M < \infty$. Defining the state as $x = \begin{pmatrix}\rho\frac{\partial w}{\partial t}\\ \frac{\partial w}{\partial \zeta}\end{pmatrix}$, the decomposition into the port-Hamiltonian form (2) is then given with Hamiltonian map and operator

$$\mathcal{H}(\zeta) = \begin{pmatrix}\frac{1}{\rho(\zeta)} & 0\\ 0 & T(\zeta)\end{pmatrix}, \qquad \mathcal{J} = \frac{\partial}{\partial\zeta}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and the dissipative interconnection and closure relation

$$\mathcal{G}_R = \frac{\partial}{\partial\zeta}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathcal{G}_R^* = -\begin{pmatrix}1 & 0\end{pmatrix}\frac{\partial}{\partial\zeta}, \qquad S = k_s.$$

In Example 3.2 we show that the p.d.e. (7) with homogeneous boundary conditions possesses a unique solution which is continuously depending on the initial condition. This result may be obtained by using standard methods, however we derive it as the consequence of the decomposition of the operator (2) into (6). We show that if the interconnection structure operator $\mathcal{J}_{\text{ext}}$, see (4), with the proper domain generates a contraction semigroup, then so does the entire class of operators $\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*$ for which the damping satisfies $S + S^* \ge \varepsilon I > 0$, see Theorem 2.2 and Example 3.2. Hence we show existence and uniqueness of solutions for all realistic damping models. For instance, a damping operator like $(S(e))(\zeta) = \int_a^b k(z,\zeta)e(z)dz$ with $k(\cdot,\cdot) \ge \varepsilon > 0$ is possible as well.

Thus our approach gives a general existence and uniqueness result for a class of systems generated by the interconnection structure operator. In Example 1.1, this operator expresses that one considers a vibrating string with structural damping depending on the strain velocity.

Note that although the decomposition of the operator (2) with the closure relation (5) looks like a feedback interconnection in which $\mathcal{J}$ is the main operator, $\mathcal{G}_R$ is the input operator, and $\mathcal{G}_R^*$ is the output operator, the approach presented here is more general. Indeed, in Example 3.1 these three operators do not form a system, but still our result is valid.

For homogeneous p.d.e.'s many techniques are possible for showing existence and uniqueness of solutions. However, for inhomogeneous p.d.e.'s, i.e., systems described by p.d.e.'s, much less techniques are known. The technique presented here can also be applied to a sub-class of systems, namely the impedance passive systems. That are systems which satisfy the following inequality along their solutions

$$\|x(t)\|^2 - \|x(0)\|^2 \le \int_0^t \langle y(\tau), u(\tau)\rangle d\tau.$$

Here x is the state, and u and y are the input and output, respectively. The closure relation S applied to such a system gives a new impedance passive system, provided $S + S^* \ge \varepsilon I > 0$.

The organisation of the paper is as follows. In the next section we present the fundamental result which, in Section 3, we first apply to homogeneous partial differential equations, illustrating that knowing the existence of solutions for a simple hyperbolic p.d.e. can give existence of solutions for seemingly unrelated p.d.e.'s. In Section 4 we show how the same fundamental result, Theorem 2.2, can be applied to study feedback for impedance passive systems. The proof of Theorem 2.2 is given in the Appendix. The discussion on extensions of the presented results can be found in the conclusion. Although there are many papers, books, etc. showing that a given p.d.e. is associated to a semigroup, the technique as presented here is new. A related technique is the internal Cayley transform for impedance passive systems of Staffans and Weiss [4,5]. In [6] it is shown that this technique corresponds to ours with the closure relation $S = I$.

By $\mathcal{L}(X)$ we denote the set of linear, bounded operators from the Hilbert space X to itself.

2. Contraction semigroups and dissipativity for a class of operators generated by composition

In this section we present some technical results on contraction semigroups for classes of operators generated by a composition of operators inspired by (6). Contraction semigroups are strongly continuous semigroups whose norm is uniformly bounded by one, see e.g. [1,7,8] for more details.

In order to simplify the presentation, we rewrite the operators of the physically motivated decomposition (4), (5), and (6). Therefore we introduce the following operator defined on the product space of two Hilbert spaces E_1 and E_2:

$$A_{\text{ext}} = \begin{pmatrix} A_1 \\ A_{21}\ 0\end{pmatrix} \tag{8}$$

with A_1 a linear operator defined on $E_1 \oplus E_2$ and A_{21} a linear operator defined on E_1. The domain of this operator is given by

$$D(A_{\text{ext}}) = \{\,(e_1,e_2) \in E_1 \oplus E_2 \mid e_1 \in D(A_{21}) \text{ and } (e_1,e_2) \in D(A_1)\}. \tag{9}$$

For this A_{ext} and an $S \in \mathcal{L}(E_2)$ we define the operator A_S on E_1 as

$$A_S e_1 = A_1\begin{pmatrix} e_1 \\ S A_{21} e_1\end{pmatrix} \tag{10}$$

with domain

$$D(A_S) = \left\{ e_1 \in D(A_{21}) \mid \begin{pmatrix} e_1 \\ S A_{21} e_1\end{pmatrix} \in D(A_1)\right\}. \tag{11}$$

Motivated by the question as formulated in the introduction, we want to derive conditions, such that A_S generates a contraction semigroup on E_1.

On Hilbert spaces, it is easy to show that any generator of a contraction semigroup will be *dissipative*, i.e., if for all x in the domain of A, $D(A)$, there holds

$$\langle Ax, x\rangle + \langle x, Ax\rangle \le 0. \tag{12}$$

Lemma 2.1. *Let A_{ext} be a dissipative operator and let $S \in \mathcal{L}(E_2)$ be accretive, i.e.*

$$S + S^* \ge 0. \tag{13}$$

Then the operator A_S as defined by (10) and (11) is dissipative.

The proof can be found in the Appendix. Note that (13) is equivalent with $\text{Re}\langle Se_2, e_2\rangle \ge 0$ for all $e_2 \in E_2$. For A_S to generate a contraction semigroup on E_1 we need a stronger condition on S. Again its proof can be found in the Appendix.