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Integral control on Lie groups

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ABSTRACT

In this paper, we extend the popular integral control technique to systems evolving on Lie groups. More explicitly, we provide an alternative definition of "integral action" for proportional(–derivative)-controlled systems whose configuration evolves on a nonlinear space, where configuration errors cannot be simply added up to compute a definite integral. We then prove that the proposed integral control allows to cancel the drift induced by a constant bias in both first order (velocity) and second order (torque) control inputs for fully actuated systems evolving on abstract Lie groups. We illustrate the approach by 3-dimensional motion control applications.

1. Introduction

Exploiting the Lie group structure of rigid body motion to model robot configuration goes back to the Denavit–Hartenberg framework and its use for robotic arms [1]. Nowadays, the Lie group viewpoint has allowed to design common control methods for various mobile robot applications including satellite attitudes [2–4], planar vehicles [5,6], submarines [7,8], surface vessels [9,10], quadrotor UAVs [11], and their coordination [5,2,12,3]. Lie groups involve a nonlinear configuration manifold where physical positions evolve, but with additional structure implying an almost linear viewpoint on the tangent bundle, where physical velocities evolve. The nonlinearity requires to adapt classical tracking and observer control tools. For example, a command proportional to configuration error must be defined as the gradient of an error function based on the distance-to-target along the manifold. The Lie group structure allows to systematically construct error functions from the relative configuration between system and target, e.g. $\phi = \frac{1}{2}\mathrm{tr}(I_{3\times 3} - Q_{\text{system}}^T Q_{\text{target}})$ for $Q_{\text{target}}, Q_{\text{system}}$ three-dimensional rotation matrices [8,13]. It also allows a canonical counterpart of Derivative control [8]. However, in an attempt to generalize the Proportional–Integral (PI) and Proportional–Integral–Derivative (PID) controllers widely used for linear(ized) industrial control applications, the nonlinearity implies more fundamental issues for the integral control term. Indeed, simply integrating objects that belong to a manifold makes mathematically no sense, e.g. a sum of rotation matrices gives in general an arbitrary square matrix of questionable use. Local linearization (retraction into a vector space) always allows a standard PI(D) control to be set up. This suggests that a proper extension might more globally recover the beneficial effect of integral control: rejecting with zero residual error a constant bias. The present paper proposes one way to extend PI(D) control to manifolds, and investigates more specifically how this rejects constant input biases on Lie groups.

In another recent approach, observers on Lie groups have been developed [13,14] and applied to the estimation of bias in measurements [15]. The observer can also be used to estimate and compensate a bias in control commands. As in the linear case, the observer approach allows more accurate performance tuning, while the PID approach requires less model knowledge.

While this work was under review, the authors became aware of independent and concurrent work [16] which the reader may want to consult for a complementary viewpoint.

2. Preliminaries and notation

2.1. Dynamical systems on manifolds and Lie groups

Let $c(t)$ be the configuration at time t of a system evolving on a nonlinear manifold $\mathcal{M}$ of finite dimension d. Its velocity $\dot{c} = \frac{dc}{dt}$ belongs to the tangent space to $\mathcal{M}$ at $c(t)$, which is a d-dimensional

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vector space $T_c\mathcal{M}$. The collection of such parameterized tangent spaces constitutes the tangent bundle $T\mathcal{M}$, a $2d$-dimensional manifold. The tangent space $T_{(c,\dot{c})}T\mathcal{M}$ to $T\mathcal{M}$ at $(c, \dot{c})$ is a vector space which, under canonical projection, contains the acceleration[1] of the system on $\mathcal{M}$. A smooth vector field on $T\mathcal{M}$ (respectively on the acceleration-part of $TT\mathcal{M}$) defines a first-order (respectively second-order) system on $\mathcal{M}$ with well-defined integrated solution. In contrast, there is no intrinsic definition of what it would mean to mathematically integrate a position error which would be a function $c : \mathbb{R} \to \mathcal{M} : t \to c(t)$ over $t \in \mathbb{R}$. For simplicity we identify tangent with cotangent space and let $\cdot$ be the scalar product between two vectors of $T_c\mathcal{M}$. The gradient $\text{grad}_c\phi \in T_c\mathcal{M}$ of $\phi : \mathcal{M} \to \mathbb{R}$ is defined such that $\text{grad}_c\phi \cdot v = \frac{d}{dt}\phi$ if $\frac{d}{dt}c = v$, for any $v \in T_c\mathcal{M}$. An element $v_1 \in T_{c_1}\mathcal{M}$ can be mapped to $v_2 \in T_{c_2}\mathcal{M}$ by a linear *transport map*. The latter depends on a trajectory from c_1 to c_2, for which there are in general several canonical choices. The differential of a transport map on $T\mathcal{M}$ is an element of the acceleration class $TT\mathcal{M}$. The transport map is needed to compare tangent vectors (i.e. velocities, accelerations) at different configurations.

A Lie group G is a smooth manifold with a group structure: a multiplication of $g, h \in G$ such that $g \cdot h \in G$, and an inverse g^{-1} with respect to a particular $e \in G$ called identity, such that $g^{-1} \cdot g = g \cdot g^{-1} = e$. We denote the typical configuration on a group by g instead of c. Lie groups feature canonical transport maps from T_gG for any $g \in G$, to $T_eG \cong \mathfrak{g}$ the Lie algebra. The left-action transport map defines a left-invariant velocity $\xi^l = L_{g^{-1}} \frac{d}{dt}g$ and the right-action transport map a right-invariant velocity $\xi^r = R_{g^{-1}} \frac{d}{dt}g$. In practice, $\xi^l \in \mathfrak{g}$ and $\xi^r \in \mathfrak{g}$ often model the velocity expressed respectively in body frame and in inertial frame (although the correspondence is not always rigorous). Then left-invariant and right-invariant accelerations $\frac{d}{dt}\xi^l$ and $\frac{d}{dt}\xi^r$ can be defined on $\mathfrak{g}$ like for vector spaces. The *adjoint representation* Ad_g is a linear g-dependent operator on the Lie algebra defined by $\xi^r = Ad_g\, \xi^l$ for any dg/dt. We have $Ad_{g^{-1}} = Ad_g^{-1}$, and $\frac{d}{dt}(Ad_g^{-1})\chi^r = [\xi^l, Ad_g^{-1}\chi^r]$ for any constant $\chi^r \in \mathfrak{g}$ if g moves according to $\xi^l = L_{g^{-1}} \frac{d}{dt}g$. Here we have introduced the Lie bracket, with property $[\xi_1, \xi_2] = -[\xi_2, \xi_1] \in \mathfrak{g}$ for all $\xi_1, \xi_2 \in \mathfrak{g}$. The gradient follows the dual mapping, e.g. we note $\text{grad}^r\phi = Ad^*_{g^{-1}}\text{grad}^l\phi$ which indeed gives $\xi^r \cdot \text{grad}^r\phi = \xi^l \cdot \text{grad}^l\phi$. An important class of groups are compact groups with unitary adjoint representation, for which $Ad_g^* = Ad_{g^{-1}}$ or equivalently $[\xi_1, \xi_2] \cdot \xi_1 = 0$ for all $\xi_1, \xi_2 \in \mathfrak{g}$.

Example SO(3). We represent the group of 3-dimensional rotations by g a rotation matrix, group operations being the matrix counterparts, and L_g the left matrix multiplication by g of $\xi^l = [\omega^l]^\wedge$ a skew symmetric matrix in $\mathfrak{g} = \{S \in \mathbb{R}^{3\times3} : S^T = -S\}$. The notation

$$\xi^l = [\omega^l]^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \Leftrightarrow [\xi^l]^\vee = \omega^l = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

interprets ω^l as the angular velocity in body frame, $\omega^r = g\,\omega^l$ the angular velocity in inertial frame. For any matrix group, $\xi^r = g\xi^l g^{-1}$ and $[\xi_a^l, \xi_b^l] = \xi_a^l\xi_b^l - \xi_b^l\xi_a^l$.

Example SE(3). The group of 3-dimensional rotations and translations is represented by

$$g = \begin{bmatrix} R & p \\ 0_{1\times3} & 1 \end{bmatrix}$$

with $R \in SO(3)$ a rotation matrix and $p \in \mathbb{R}^3$ a translation vector. The group operations become matrix operations as for $SO(3)$, the elements of the Lie algebra write

$$\xi^l = g^{-1}\frac{d}{dt}g = \begin{bmatrix} [\omega^l]^\wedge & v^l \\ 0_{1\times3} & 0 \end{bmatrix}$$

with v^l the translation velocity expressed in body frame. The group $SE(3)$ is not compact and hence its adjoint representation $Ad_g\xi^l = g\xi^l g^{-1}$ is not unitary: a large left-invariant velocity does not correspond to a large right-invariant velocity, and vice versa.

2.2. Proportional and PD control on Lie groups

PD controllers on manifolds and Lie groups have been previously proposed, see Introduction. Following a simplified version of [8], we define an error function $\phi(r^{-1}g)$ between current configuration $g(t)$ and target configuration $r(t)$. We make the typical assumption that $\phi(h)$ increases with the distance from h to identity e, has a single local minimum $\phi(e) = 0$ at the target, possibly (unavoidably on compact Lie groups) a set of other critical points.

For simplicity we assume r to be fixed; feedforward can easily account for a moving $r(t)$, e.g. by adding a term $\xi^l_{ff} = Ad_{g^{-1}r}\chi^l$ to the velocity command if $\frac{d}{dt}r = r\,\chi^l$. In a first-order system,

$$\xi^l_p = -k_P\text{grad}^l\phi$$

is viewed as a *proportional* feedback term. For a well-chosen ϕ, the linearization of ξ^l_p shall indeed be like proportional control for $r^{-1}g \simeq e$. In a second-order system

$$L_{g^{-1}}\frac{d}{dt}g = \xi^l, \quad \frac{d}{dt}\xi^l = F^l$$

with input torque/force F^l, the *proportional* control is

$$F^l_p = -k_P\text{grad}^l\phi$$

and the *derivative* control term is

$$F^l_d = -k_D\xi^l$$

(slightly more involved if r was time-varying). A basic result of e.g. [8, Theorem 4.6] is that for fully actuated systems, both the first-order system with P-control and the second-order system with PD-control converge to the target, according to a Lyapunov function built around ϕ.

In the following we show how to add integral control to this setting and recover this perfect convergence in presence of a constant input bias. In relation with this, we note that on Lie groups, a strong enough bias might not only prevent convergence close to the equilibrium, but even drive the system into a periodic motion. This is exemplified on the N-torus by weakly coupled Kuramoto oscillators with different natural frequencies [17].

3. A definition of integral control in the PI / PID context

In this section, we propose a general definition of integral control in the context of proportional or proportional–derivative control on nonlinear manifolds. In the next section, we specialize to Lie groups and prove how the proposed integral control allows to cancel the negative effect of constant biases. We propose a simple *intrinsic* way to define the integral control term on nonlinear manifolds, where the configuration error cannot be integrated:

Definition 1a. The integral term u_I for PI (respectively PID) control on a manifold is obtained as the integral of the P (respectively PD) control command u_P (respectively u_{PD}).

The spirit of this definition is to integrate the effort that the controller has been making so far. On a vector space, it is equivalent to the traditional definition as an integral of the output error. Indeed, we have:

[1] Note that we are not speaking about Euler–Lagrange systems and possible curvature-induced accelerations here, we just define the spaces on which we work.