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Extensions of weak-invariance principle for nonlinear switched systems with time-invariant and time-varying subsystems

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ABSTRACT

This paper addresses the stability problem for nonlinear switched systems with both time-invariant and time-varying subsystems. Given that Lyapunov-like functions are difficult to construct for nonlinear switched systems, generalized invariance principles are established based on observer functions. For nonlinear switched systems where the subsystems share Lyapunov-like functions, the generalized invariance principles can be specialized to Lyapunov-based invariance principles, where accurate convergent region can be obtained. In addition to the above efforts, the definitions of p-limit system and limit system set are presented, under which the proposed invariance principles are extended to nonlinear switched systems with time-varying subsystems. Illustrative examples show the effectiveness of the theoretical results.

1. Introduction

Switched systems exhibit complicated dynamical behaviours under logical rule orchestrating the active subsystems at discontinuous time constants. Typical examples of switched systems include buck–boost converters [1], direct injection stratified charge engines [2], and consensus of multi-agent systems with switching topologies [3,4].

Extensions of invariance principle have been developed in recent years. In [5], extensions of the invariance principle relative to dwell time switched systems and constrained switched systems were proposed. In [6], under observability assumptions or common bounds on the Lyapunov-like functions, invariance principles for switched systems were developed via hybrid systems techniques. In [7], union weak-invariance principle and intersection weak-invariance principle were proposed, where the construction of Lyapunov functions was not needed. Other results on invariance principles can be found in [8,9].

Although some extensions of invariance principle for stability of switched systems have been proposed, there are still three problems need to be further considered. First, so far almost all the extensions of invariance principle relies on the existence of Lyapunov-like functions. The key restriction of these Lyapunov-like functions is the nonincreasing requirement over the time intervals and discontinuous switching times. However, there do not exist general methods on how to find such Lyapunov-like functions for switched systems with time-varying subsystems. Second, no accurate convergent region can be got for invariance principles of general nonlinear switched systems. It was shown in the literature (see, e.g., [5]) that under certain assumptions, every switched solution was attracted to the largest weakly invariance set in the union of the zero loci of every piecewise Lyapunov-like functions. Usually, the union set is unbounded, and therefore accurate convergent region cannot be obtained. Third, there are few investigations on switched systems with time-varying subsystems.

Motivated by the above mentioned considerations, this paper aims to propose effective extensions of invariance principle for stability problems of switched systems with both time-invariant and time-varying subsystems. The main contribution of this paper is threefold:

- For nonlinear switched systems with time-invariant subsystems, where Lyapunov-like functions are difficult to obtain, generalized invariance principles are established based on only observer functions. These generalized invariance principles also include as special cases the integral invariance principle [10,11].
- For nonlinear switched systems with time-invariant subsystems, where Lyapunov-like functions exist, Lyapunov-based weak-invariance principles are established, under which more accurate convergent region is obtained compared with the results in [5,12].
- The definitions of p-limit system and limit system set are presented, based on which the results are extended to nonlinear switched systems with time-varying subsystems. Also, multiple bounded functions are presented, which can be used to judge state boundedness of general nonlinear switched systems.

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2. Preliminaries

In this paper, we consider the following switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$
$$y(t) = h_{\sigma(t)}(x(t)) \quad (1)$$

where $\sigma(t) : [0, \infty) \to \mathcal{P} = \{1, 2, \ldots, N\}$ is a piecewise constant, right-continuous function, called a switching signal. For $\forall p \in \mathcal{P}$, $f_p(x)$ and $h_p(x)$ are Lipschitz continuous vector fields of $\mathbb{R}^n$ with $f_p(0) = 0$ and $h_p(0) = 0$. The time instants $\{\tau_i : i \in \mathbb{N}\}$ at which $\sigma(t)$ is discontinuous are called the switching times. A continuous piecewise-smooth curve $x(t) : \mathcal{I} \to \mathbb{R}^n$ is a solution of system (1) in the time interval $\mathcal{I}$ if $\dot{x}(t) = f_{\sigma(t)}(x(t))$ for all $t \in [\tau_i, \tau_{i+1}) \cap \mathcal{I}$. In this paper, we investigate forward complete solutions, i.e., $\mathcal{I} = [0, \infty)$, and we denote $x(t, x_0)$ as a forward complete solution of (1) with initial value x_0. For each $p \in \mathcal{P}$, let $\sigma|p = \{\tau_{p_1}, \tau_{p_1+1}, \tau_{p_2}, \tau_{p_2+1}, \ldots\}$ be the sequence of switching times when subsystem p is switched on or switched off, and $\Phi_p = \cup_{i\in\mathbb{N}}[\tau_{p_i}, \tau_{p_i+1})$ be the union of intervals on which subsystem p is active.

Assumption 1. Switched system (1) has a nonvanishing dwell time, i.e., there exists a positive constant τ such that

$$\inf_i(\tau_{i+1} - \tau_i) \geq \tau \quad (2)$$

where $\{\tau_i : i \in \mathbb{N}\}$ is the sequence of switching times associated with $\sigma(t)$. The set of all switched solutions possessing a nonvanishing dwell time is denoted by $\mathcal{S}_{dwell}$.

Assumption 2. There exists $\bar{T} > 0$ such that for $\forall \bar{t} \geq 0$, $\sigma(t)$ satisfies $\sigma(t) \not\equiv \sigma(\bar{t})$ during $[\bar{t}, \bar{t} + \bar{T}]$.

Remark 1. Assumption 2 presents an essential characteristic for switched systems. If Assumption 2 does not hold, then there exists $\bar{t} > 0$ such that for $\forall \bar{T} > 0$, the switching signal $\sigma(t)$ satisfies $\sigma(t) \equiv \sigma(\bar{t})$ during $[\bar{t}, \bar{t} + \bar{T}]$, which implies $\sigma(t)$ will be constant for $\forall t \geq \bar{t}$. Therefore, the switched systems will become "non-switching", which is trivial in this paper. The switching signals under Assumption 2 include those studied in [3,4,13].

Definition 1 (*[14]*). A function $y(t) : [0, \infty) \to \mathbb{R}$ is said to be weakly meagre if $\lim_{i\to\infty}(\inf_{t\in I_i} \|y(t)\|) = 0$ for any disjoint closed intervals $\{I_i : i \in \mathbb{N}\}$ with $\inf_{i\in\mathbb{N}} \mu(I_i) > 0$, where $\mu(I_i)$ is the Lebesgue measure of the interval I_i.

Definition 2 (*[15]*). Let $x(t, x_0) : [0, \infty) \to \mathbb{R}^n$ be a continuous curve. A point $a \in \mathbb{R}^n$ is an ω-limit point of $x(t, x_0)$ if there is a divergent sequence $\{t_i : i \in \mathbb{N}\}$ such that $\lim_{i\to\infty} x(t_i, x_0) = a$. The set of all ω-limit points of $x(t, x_0)$ is denoted by $\omega^+(x_0)$.

Definition 3 (*[15]*). A continuous curve $x(t, x_0)$ is attracted to a compact set $\mathcal{M}$ if for each $\epsilon > 0$ there exists a time point $T > 0$ such that $x(t, x_0) \in \mathcal{B}(\mathcal{M}, \epsilon)$ for $t \geq T$, where $\mathcal{B}(x, \epsilon)$ is the open ball of radius ϵ centred at x and $\mathcal{B}(\mathcal{M}, \epsilon) = \cup_{x\in\mathcal{M}} \mathcal{B}(x, \epsilon)$. Clearly, $x(t, x_0)$ is attracted to $\mathcal{M}$ if and only if $\lim_{t\to\infty} \text{dist}(x(t, x_0), \mathcal{M}) = 0$.

Definition 4 (*[5]*). For $\forall p \in \mathcal{P}$. We say that a function $\psi(t)$ satisfies property (p) on an interval $[\gamma, \delta]$ if $\psi(t)$ is a solution of $\dot{x} = f_p(x)$ on $[\gamma, \delta]$.

Definition 5 (*[5]*). We say that a compact set $\mathcal{W}$ is weakly invariant with respect to (1) if for $\forall a \in \mathcal{W}$, there exist an index $p \in \mathcal{P}$ and a real number $b > 0$ such that the solution $x(t, x_0)$ of $\dot{x} = f_p(x)$ with initial value $x_0 = a$ satisfies $x(t, x_0) \in \mathcal{W}$ for either $t \in [-b, 0]$ or $[0, b]$.

Lemma 1. *Let $x(t, x_0)$ be a bounded solution of switched system (1) and τ be the dwell time presented in Assumption 1. Then for each point $a \in \omega^+(x_0)$, there exist an index $p' \in \mathcal{P}$, an interval $[\gamma, \delta]$ $(0 \in [\gamma, \delta], \delta - \gamma \geq \tau/2)$ and a divergent sequence $\{t_i : i \in \mathbb{N}\}$ such that*

1. *$\sigma(t + t_i) = p'$ for $\forall t \in [\gamma, \delta]$ and $\forall i \in \mathbb{N}$;*
2. *$\psi_i(t) := x(t + t_i, x_0)$ satisfy property (p') on $[\gamma, \delta]$ for $\forall i \in \mathbb{N}$ and $\lim_{i\to\infty} \psi_i(0) = a$.*

Proof. See Proposition 2 of [5].

Lemma 2 (*[12]*). *Let $x(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded solution of (1). Then, $\omega^+(x_0)$ is nonempty, compact and weakly invariant. Moreover, $x(t, x_0)$ is attracted to $\omega^+(x_0)$.*

Lemma 3 (*[15]*). *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and nonnegative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies $y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$ for $a \leq t \leq b$, then on the same interval, $y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left(\int_s^t \mu(\tau) d\tau\right)ds$. In particular, if $\lambda(t) \equiv \lambda$ is a constant, then $y(t) \leq \lambda \exp\left(\int_a^t \mu(\tau)d\tau\right)$.*

3. Switched systems (time-invariant subsystems)

3.1. Generalized invariance principles

In this subsection, generalized invariance principles for system (1) are developed based on observer functions.

Lemma 4. *Suppose that Assumption 1 holds. Let $x(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded solution of the nonlinear switched system (1) and $\omega^+(x_0)$ be the limit set with respect to $x(t, x_0)$. If $y(t)$ is weakly meagre, then we have that $\omega^+(x_0) \subseteq \cup_{p\in\mathcal{P}}\{x : h_p(x) = 0\}$.*

Proof. By Lemma 1, we have that for $\forall a \in \omega^+(x_0)$, there exist $p' \in \mathcal{P}$, an interval $[\gamma, \delta]$ $(0 \in [\gamma, \delta], \delta - \gamma \geq \tau/2)$ and a sequence $\{\psi_i(t) = x(t + t_i, x_0) : i \in \mathbb{N}\}$ such that

$$\dot{\psi}_i(t) = f_{p'}(\psi_i(t)), \quad \forall t \in [\gamma, \delta] \quad (3)$$

where $\lim_{i\to\infty} \psi_i(0) = a$. Therefore, we can obtain that

$$\psi_i(t) = \psi_i(0) + \int_0^t f_{p'}(\psi_i(s))ds, \quad \forall t \in [\gamma, \delta]. \quad (4)$$

Let $l_{p'} > 0$ be the Lipschitz constant such that $\|f_{p'}(x) - f_{p'}(y)\| \leq l_{p'}\|x - y\|$ for $\forall x, y \in \mathbb{R}^n$. Since $\lim_{i\to\infty} \psi_i(0) = a$, then for $\forall \epsilon > 0$, there exists $K > 0$ such that $\forall i, j > K$

$$\|\psi_i(0) - \psi_j(0)\| < \epsilon \exp\left(-l_{p'}(\delta - \gamma)\right). \quad (5)$$

By Lemma 3, we obtain that for $\forall t \in [\gamma, \delta]$

$$\begin{aligned} &\|\psi_i(t) - \psi_j(t)\| \\ &\quad \leq \|\psi_i(0) - \psi_j(0)\| + \left\| \int_0^t \left(f_{p'}(\psi_i(s)) - f_{p'}(\psi_j(s))\right) ds \right\| \\ &\quad \leq \epsilon \exp(-l_{p'}(\delta - \gamma)) + l_{p'} \int_0^t \|\psi_i(s) - \psi_j(s)\| \, ds \\ &\quad \leq \epsilon \exp(-l_{p'}(\delta - \gamma)) \exp(l_{p'}(\delta - \gamma)) < \epsilon. \end{aligned} \quad (6)$$

Therefore, we conclude that $\{\psi_i(t) : i \in \mathbb{N}\}$ uniformly converges to a continuous function $\psi(t)$ on the interval $[\gamma, \delta]$, where $\psi(0) = \lim_{i\to\infty} \psi_i(0) = a$. Next, we will show that $a \in \cup_{p\in\mathcal{P}}\{x : h_p(x) = 0\}$. Suppose to the contrary that $a \notin \cup_{p\in\mathcal{P}}\{x : h_p(x) = 0\}$. Then, we can define $\rho = \min_{p\in\mathcal{P}} \|h_p(a)\| > 0$. Since $h_{p'}(\psi(t))$ is continuous with respect to t and $h_{p'}(\psi(0)) = h_{p'}(a) \neq 0$, we