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Control and stabilization for the wave equation with variable coefficients in domains with moving boundary

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ABSTRACT

The study of stabilization and control for PDEs with variable coefficients involves higher level of complexity than the corresponding case of constant coefficients. Further compounding the complexity are the concern and effects of dynamic boundary motion. The problem in its general form is extremely challenging to treat, but under certain specific physical and geometric conditions, such as the time-likeness of the boundary and a limited speed of domain expansion, energy decay estimates can be established and the exact controllability can also be obtained by control-theoretic and Riemannian-geometric methods. Our approach here is based on the Bochner technique of differential geometry in terms of Riemannian metric and geometric multipliers, by generalizing an energy identity method used earlier in Bardos and Chen (1981). Concrete examples are also given to illustrate the geometric conditions and the theorems.

1. Introduction

Let $\Omega(t)$ be bounded open sets in $\mathbb{R}^n$ with smooth boundary $\partial\Omega(t)$ for each $t > 0$. For $0 \le t_1 < t_2$, let

$$Q(t_1, t_2) = \bigcup_{t=t_1}^{t_2} \Omega(t) \times \{t\} \quad \text{and} \quad \Sigma(t_1, t_2) = \bigcup_{t=t_1}^{t_2} \partial\Omega(t) \times \{t\}$$

denote the spatiotemporal domain and the lateral surface from t_1 to t_2, respectively. We assume that $\Sigma(t_1, t_2)$ is piecewise smooth. In particular, we denote $Q = Q(0, \infty)$ and $\Sigma = \Sigma(0, \infty)$. Consider the following initial–boundary value problem

$$\begin{cases} u_{tt} - \operatorname{div}(A(x)\nabla u) = 0, & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in \Sigma, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega(0). \end{cases} \tag{1.1}$$

Here $A(x) = (a_{ij}(x))$ are symmetric and positive definite matrices for all $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. In general, the parameterization of Σ depends on the variable t. System (1.1) is said to be a problem posed on a domain with moving boundary. We are interested in the stabilization and control of system (1.1).

When $A(x) \equiv I$, the identity matrix, the PDE in (1.1) is the standard wave equation. In this case, the moving boundary problem has been studied; see e.g., [1–5], and the references therein. Under certain assumptions on the moving boundary, [1] considered the exact controllability and stabilization of problem (1.1). Using a suitable change of variables, [6] studied the existence and asymptotic behavior of global regular solutions of the mixed problem for the Kirchhoff nonlinear model. Also [7] considered a damped Klein–Gordon equation in a non-cylindrical domain and obtained the existence of global solutions and the exponential decay of energy.

But if the medium of vibration is nonhomogeneous, or if general curvilinear coordinates are used, then the matrix $A(x)$ will not be I. Rather, entries $a_{ij}(x)$ of $A(x)$ become functions of x. This constitutes a far more complex and challenging problem, and more complicated tools are needed. A natural one to consider is the Riemannian-geometric method. The Riemannian-geometric method was first introduced in [8] for the controllability of the wave equation with variable coefficients and later generalized and extended in [9–16,8,17–23], etc. For the control problems of partial differential equations this approach shows many advantages. A key idea therein is the Bochner technique that provides great simplification in the derivation of the energy multiplier identities, which

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are needed for uniqueness and observability inequalities. Furthermore the curvature theory yields global geometric information on controllability/stabilization properties for the variable coefficient models [24,22,25].

Here a major contribution of this paper is the synthesis of [1,8] for the study of the stabilization and the exact controllability of problem (1.1). To the best of our knowledge, this is the first paper to study the variable coefficient problems with moving boundary, where we demonstrate that geometric multipliers are also effective for problems with time-varying domains.

This paper is organized as follows.

Section 2 provides some prerequisites in differential geometry, needed assumptions, and the statements of the Main Theorems 2.1–2.3, with two concrete examples.

Section 3 contains four subsections. The first three subsections establish the proofs, in sequential order, for Theorems 2.1–2.3, while in the very last Section 3.4, the proof of the technical Proposition 2.1 is furnished.

Brief concluding remarks are given at the very end.

2. Differential-geometric preliminaries and statements of the main theorems

We now proceed to introduce notation and then state our main results. Let

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \tag{2.1}$$

be a Riemannian metric on $\mathbb{R}^n$ and regard the pair $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the Riemannian metric g induces an inner product and norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X,\ Y\rangle, \qquad |X|_g^2 = \langle X, X\rangle_g, \quad X,\ Y \in \mathbb{R}^n, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in the Euclidean space $\mathbb{R}^n$. For any $w \in H^1(\mathbb{R}^n)$, where $H^s(\mathbb{R}^n)$ denote the usual Sobolev space of order s, define

$$|\nabla_g w|_g^2 = \sum_{i,j=1}^n a_{ij}(x) w_{x_i}(x) w_{x_j}(x) \quad \text{for } x \in \mathbb{R}^n, \tag{2.3}$$

where ∇_g is the gradient with respect to the Riemannian metric g.

Let $\boldsymbol{R}$ be the curvature tensor of the metric g; $\boldsymbol{R}$ is a fourth order tensor field on $\mathbb{R}^n$, see [22] or other references on Riemannian geometry. Denote $\gamma(t)$ a normal geodesic of the metric g initiating at the origin 0. The radial Ricci curvature is given by

$$\mathrm{Ric}\,(\gamma(t)) = \sum_{i=2}^n \boldsymbol{R}(\dot\gamma(t), e_i, \dot\gamma(t), e_i),$$

where $\dot\gamma(t), e_2, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n_{\gamma(t)}$ for each $t \geq 0$. If $A(x) = I$ is the unit metric in $\mathbb{R}^n$, then $\boldsymbol{R} = 0$ and $\mathrm{Ric}\,(\gamma(t)) = 0$ for all $t \geq 0$.

2.1. Stabilization and growth estimates

Let u be a solution to (1.1). We define the energy of (1.1) as

$$E(t) = \frac{1}{2}\int_{\Omega(t)} (u_t^2 + |\nabla_g u|_g^2)dx, \tag{2.4}$$

where $|\nabla_g u|_g^2$ is given by (2.3).

In the case of the standard wave equation ($A(x) = I$), under certain assumptions, [1] obtained the following estimate

$$E(t) \leq \frac{c}{t}E(0) \quad \text{for } t \text{ large}, \tag{2.5}$$

for some constant $c > 0$.

We seek suitable geometric conditions under which an estimate (2.5) also holds for (1.1).

Let $\rho(x) = d(x, 0)$ be the distance function from $x \in \mathbb{R}^n$ to the origin $0 \in \mathbb{R}^n$ in the metric g, given by (2.1). In our analysis this distance function ρ will play a key role. If $A(x) = I$, then

$$\rho(x) = |x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

For a general metric (2.1), the structure of the distance function ρ is more complex, see [22] or other references on Riemannian geometry. Define

$$f(x) = \mathrm{div}(A(x)\nabla\rho) \quad \text{for } x \in \mathbb{R}^n. \tag{2.6}$$

In addition, let $\nu = (\nu_1, \ldots, \nu_n, \nu_t) = (\nu_x, \nu_t)$ be the unit outward normal at (x, t) on $\Sigma(t_1, t_2)$ in $\mathbb{R}^n \times \mathbb{R}$. Then

$$|\nu_x|^2 + \nu_t^2 = 1 \quad \text{for}(x, t) \in \Sigma(t_1, t_2).$$

We make the following assumptions:

(H_1) (time-likeness of Σ) $|\nu_t| < |A\nu_x|_g$ on Σ;

(H_2) For $t \geq 0$, the domain $\Omega(t)$ is expanding, i.e.,

$$\Omega(t_1) \subset \Omega(t_2) \quad \text{for } 0 \leq t_1 \leq t_2.$$

(H_3) There exist $T_0 > 0$ and $\theta : 0 < \theta < 1$ such that $\Omega(t) \subset \{x \in \mathbb{R}^n | \rho(x) < \theta(t + T_0)\}$ and

$$(t + T_0)\nu_t + \rho A(x)\nu_x(\rho) \leq 0 \quad \text{on } \partial\Omega(t) \text{ for } t \geq 0. \tag{2.7}$$

(H_4) The function f, given by (2.6), satisfies

$$f^2(x) + \rho f(x) f_\rho(x) + 2f_\rho(x) + \rho f_{\rho\rho}(x) \leq 0 \quad \text{for } x \in \mathbb{R}^n, \tag{2.8}$$

and

$$4f(x) + \rho f^2(x) + 2\rho f_\rho(x) \geq 0 \quad \text{for } x \in \mathbb{R}^n, \tag{2.9}$$

where f_ρ and $f_{\rho\rho}$ denote the first and the second directional derivatives of f along $\nabla\rho$, respectively.

Remark 2.1. (1) Assumption (H_1) generalizes the condition $|\nu_t| < |\nu_x|$ in [1]. This condition says that Σ is "like time" so the initial–boundary value problem on the non-cylindrical domain Q is well posed. If $|\nu_t| < |A\nu_x|_q$ is violated, then generally there is lack of uniqueness.

(2) Assumption (H_3) says that the "speed" of expansion should be at most a fraction θ ($0 \leq \theta < 1$) of the wave speed, and be "somewhat uniform" in the (generalized) radial direction.

(3) Assumption (H_4) provide technical conditions needed for the proofs. □

We now have the following.

Theorem 2.1. *Let the radial Ricci curvature of* $(\mathbb{R}^n, g)$ *be nonpositive. Assume* (H_1)–(H_4). *Then*

$$E(t) \leq \frac{(1+\theta)T_0}{1-\theta}\frac{1}{t+T_0}E(0) \quad \textit{for } t \geq 0. \tag{2.10}$$

Remark 2.2. Let $A(x) = I$ be the case of the usual wave equation. Then the Ricci curvature is identically zero. Assumptions (H_1)–(H_3) were made in [1]. Since $\rho(x) = |x|$ for $x \in \mathbb{R}^n$ in this case, we have

$$f(x) = \mathrm{div}\,\nabla\rho = \frac{n-1}{\rho} \quad \text{for } x \in \mathbb{R}^n,$$

and

$$f^2(x) + \rho f(x) f_\rho(x) + 2f_\rho(x) + \rho f_{\rho\rho}(x) = 0 \quad \text{for } x \in \mathbb{R}^n, \tag{2.11}$$