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Discretisation of sparse linear systems: An optimisation approach☆

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ABSTRACT

This paper addresses the discretisation problem for sparse linear systems. Classical methods usually destroy sparsity patterns of continuous-time systems. We develop an optimisation procedure that yields the best approximation to the discrete-time dynamical matrix with a prescribed sparsity pattern and subject to stability and other constraints. By formulating this problem in an adequate manner, tools from convex optimisation can be then applied. Error bounds for the approximation are provided for special classes of matrices. Numerical examples are included.

1. Introduction

Discretisation of continuous-time linear systems is a well established procedure, due to its key role in digital control engineering [1] and sampled-data systems [2]. Nevertheless, the requirement for novel discretisation methods is still emerging in several areas. Examples of such areas include networked control systems [3] and large scale collaborative optimisation problems such as those found in intelligent transportation systems (ITS) applications [4]. The basic objective in these new application areas is that one seeks preserve a certain property of interest. In this paper, we will consider the problem of realising discretisation algorithms that preserve sparsity constraints.

Large-scale dynamical systems usually present structural characteristics, which are fundamental to describe their behaviour [5]. Indeed, these systems usually derive from the dynamical interaction of several interconnected subsystems, which can model industrial settings [6], automated highway systems (AHS) [7], structural dynamics [8] and network flow problems [9]. Thus, these structures arise not only due to physical properties of the system being modelled, but also due to communication and costs limitations. For instance, an AHS may only allow communication between neighbouring vehicles, which builds up a sparsity pattern in its continuous-time state dynamics.

The sparsity patterns presented by large scale systems are usually obtained for their continuous-time formulation. However, the discrete-time versions of these models are the ones that will be either implemented or simulated and, as it will be further discussed in the sequel, the classical discretisation methods usually destroy this sparsity pattern. To avoid this, discretisation methods based on Euler's forward approximation to the exponential can be adopted [10,11]. Unfortunately, these approximations are usually good only for small values of the sampling period.

This paper provides novel discretisation techniques for sparse linear systems. We break free from the classical approach of approximating the matrix exponential and recast the problem in the setting of convex optimisation, which can be solved efficiently with the existing methods. Error bounds are provided for special classes of sparse matrices that arise in several practical applications.

The notation is standard. Capital letters denote matrices and small letters denote vectors and scalars. For matrices and vectors, $(')$ denotes transpose and, for a block-structured symmetric matrix, $(\star)$ denotes each of its symmetric blocks. The sets of real, nonnegative real, positive real and natural numbers are indicated as $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$ and $\mathbb{N}$. For symmetric matrices, $X \succ 0$ denotes that X is positive definite. For square matrices, $\mathbf{tr}(\cdot)$ denotes the trace function and

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$\sigma_{\max}(\cdot)$ represent its maximum singular value. Block diagonal matrices are defined by its blocks using the notation **diag**$(\cdot)$, as usual. For a real matrix A, its *spectral* and its *Frobenius norms* will be denoted by $\|A\|_2 = \sigma_{\max}(A)$ and $\|A\|_F = \sqrt{\mathbf{tr}(A'A)}$. Finally, for a real function f of one variable, $f^{(n)}$ denotes its n-th order derivative.

2. Discretisation of sparse linear systems

2.1. Problem statement

In this paper, we consider a continuous-time, linear, time-invariant (LTI) autonomous system given by

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{1}$$

in which $x : \mathbb{R}_+ \to \mathbb{R}^n$ is its state. In the classical discretisation problem, the discrete-time realisation

$$x[k+1] = Mx[k], \qquad x[0] = x_0, \tag{2}$$

must be determined to ensure that $x(kh) \approx x[k]$ for all $k \in \mathbb{N}$, where $h \in \mathbb{R}_+^*$ is the *discretisation step* or *sampling period*. It is a well known fact [12] that, whenever $M = e^{hA} = \sum_{k=0}^{\infty}(hA)^k/k!$, the discrete-time LTI system (2) is such that $x(kh) = x[k]$ for all $k \in \mathbb{N}$. Hence, whenever this exact approach can be adopted, the discretisation problem is readily solved from the computation of the matrix exponential, [13]. However, in some applications, one seeks to determine M that approximates e^{hA} and preserves some specific properties, such as sparsity.

In what follows, we assume that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a sparse matrix, whose specific sparsity pattern is defined by the set $\mathcal{S} \subset \mathbb{R}^{n\times n}$. Formally, one can consider the set $\mathcal{I}_{\mathcal{S}} \subset \{1,\dots,n\}^2$ composed of pairs (i,j) such that a_{ij} is allowed to be nonzero and, therefore, define $\mathcal{S}$ as the set that contains all matrices $S = (s_{ij}) \in \mathbb{R}^{n\times n}$ such that $s_{ij} = 0$ whenever $(i,j) \notin \mathcal{I}_{\mathcal{S}}$. Due to its definition, note that $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times n}$. However, it is of interest to observe that $A \in \mathcal{S}$ does not ensure that $e^{hA} \in \mathcal{S}$ for some $h > 0$. In fact, the discretisation procedure $A \mapsto e^{hA}$ generally destroys structural properties of the original continuous-time system. This phenomenon, which ensures $x(kh) = x[k], \forall k \in \mathbb{N}$, creates direct dependencies between state variables that do not exist in the original continuous-time dynamics. Hence, considering another subspace $\mathcal{R} \subset \mathbb{R}^{n\times n}$ that defines a sparsity pattern, our main goal is to determine $M \in \mathcal{R}$ such that $M \approx e^{hA}$ for some $h > 0$ given. It is often desirable that $\mathcal{R} = \mathcal{S}$, but this may be relaxed in some situations, where we will allow $\mathcal{R} \supset \mathcal{S}$. For example, in ITS applications, local inter-vehicle communication may be possible. Moreover, $\mathcal{R}$ also presents a set $\mathcal{I}_{\mathcal{R}} \subset \{1,\dots,n\}^2$ composed of the nonzero positions allowed by its structure and, since $\mathcal{R}$ may relax some constraints imposed by $\mathcal{S}$, it follows that $\mathcal{I}_{\mathcal{R}} \supset \mathcal{I}_{\mathcal{S}}$. This property can be exploited not only to improve the quality of the approximation to the matrix exponential but also to make the optimisation feasible in some situations.

2.2. Mathematical preliminaries

The following auxiliary results and definitions are extensively used throughout. The matrix exponential can be computed via numerical methods based on *Padé approximants* [13,14]. Two particular cases of approximants to the matrix exponential are Tustin's formula

$$e^{sA} \approx T(sA) \triangleq \left(I - \frac{s}{2}A\right)^{-1}\left(I + \frac{s}{2}A\right) \tag{3}$$

and Taylor's polynomial of order λ, centred at the origin,

$$e^{sA} \approx R_\lambda(sA) \triangleq \sum_{k=0}^{\lambda}\frac{s^k}{k!}A^k. \tag{4}$$

Padé approximants are widely adopted in discretisation methods [11,15]. It is also worth pointing out that Tustin's approximant plays a key role in control theory [12].

The following theorem [13] provides an error bound for the approximation of a matrix function.

Theorem 1. *If $f(\cdot)$ has the Taylor series representation $f(z) = \sum_{k=0}^{\infty}\alpha_k z^k$ in an open disk containing the eigenvalues of $A \in \mathbb{C}^{n\times n}$, then*

$$\left\| f(A) - \sum_{k=0}^{\lambda}\alpha_k A^k \right\|_2 \leq \frac{n\|A\|_2^{\lambda+1}}{(\lambda+1)!}\max_{0\leq s\leq 1}\left\|f^{(\lambda+1)}(sA)\right\|_2. \tag{5}$$

We are particularly interested in the case $f(\cdot) \equiv \exp(\cdot)$, which implies

$$\left\| e^A - \sum_{k=0}^{\lambda}\frac{A^k}{k!} \right\|_2 \leq \frac{n\|A\|_2^{\lambda+1}}{(\lambda+1)!}e^{\|A\|_2}. \tag{6}$$

Additionally, it is also possible to obtain bounds for the Frobenius norm and for any Padé approximant to the exponential; see [13,14].

2.3. Discretisation as an optimisation problem

Now we focus on the main problem stated before, which can be analysed, in a simple manner, as a projection problem. Indeed, given a continuous-time system with realisation (1) and a step size $h \in \mathbb{R}_+^*$, we wish to determine $M^\star \in \mathcal{R} \subset \mathbb{R}^{n\times n}$ such that $M^\star$ is the "closest" element of $\mathcal{R}$ to e^{hA}, with respect to the metric δ. Thus, its general formulation is

$$M^\star = \arg\inf_{M\in\mathcal{R}} \delta\left(M, e^{hA}\right), \tag{7}$$

in which δ provides the notion of distance between the approximation M and the exact discrete-time matrix e^{hA}, for any $h \in \mathbb{R}_+^*$ given. Thus, from the computational viewpoint, it represents the *error* yielded by the approximation. Note that, whenever δ is induced by a matrix norm, the optimisation problem (7) is convex; see [16].

In this paper, we are particularly interested in two widely adopted norms in approximation problems (see [17]): the spectral norm and the Frobenius norm. For the first case, it is possible to show [18,19] that, for a given sampling period $h \in \mathbb{R}_+^*$, there exists $\sigma \in \mathbb{R}_+^*$ such that the error bound $\|M - e^{hA}\|_2 < \sigma$ holds if, and only if, the linear matrix inequality (LMI)

$$\begin{pmatrix} \sigma^2 I & \star \\ M - e^{hA} & I \end{pmatrix} \succ 0 \tag{8}$$

is satisfied. Accordingly, for the Frobenius norm case, the error bound $\|M - e^{hA}\|_F < \sigma$ holds if, and only if, there exists $W \succ 0$ such that the LMIs

$$\mathbf{tr}(W) < \sigma^2, \qquad \begin{pmatrix} W & \star \\ M - e^{hA} & I \end{pmatrix} \succ 0 \tag{9}$$

hold. Hence, whenever δ is induced by $\|\cdot\|_2$, the best approximation in $\mathcal{R}$ to the matrix exponential can be obtained solving the convex optimisation problem

$$(M^\star, \sigma^\star) = \arg\inf_{M\in\mathcal{R},\sigma}\{\sigma : (8)\}. \tag{10}$$

Similarly, for the Frobenius norm, the best approximation in $\mathcal{R}$ to the matrix exponential can be obtained solving

$$(M^\star, \sigma^\star) = \arg\inf_{M\in\mathcal{R},\sigma}\{\sigma : (9)\}. \tag{11}$$

In both cases, $\sigma^\star = \delta(M^\star, e^{hA})$ is the optimal value for the error yielded by the approximation. Additionally, both optimisation problems are convex, as expected.