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Distributed estimation of Laplacian eigenvalues via constrained consensus optimization problems

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ABSTRACT

From the recent literature, we know that some consecutive measurements of the consensus protocol can be used to compute the exact average of the initial condition. In this paper, we show that these measurements can also be used for estimating the Laplacian eigenvalues of the graph representing the network. As recently shown in the literature, by solving the factorization of the averaging matrix, the Laplacian eigenvalues can be inferred. Herein, the problem is posed as a constrained consensus problem formulated two-fold. The first formulation (direct approach) yields a non-convex optimization problem solved in a distributed way by means of the method of Lagrange multipliers. The second formulation (indirect approach) is obtained after an adequate re-parameterization. The problem is then convex and is solved by using the distributed subgradient algorithm and the alternating direction method of multipliers (ADMM). The proposed algorithms allow estimating the actual Laplacian eigenvalues with high accuracy.

1. Introduction

Networks are generally represented by means of graphs where vertices represent nodes whereas edges represent the existence of an interaction between them. Several features of a given network are captured by the graph Laplacian matrix (see [1] for a comprehensive survey on properties of the Laplacian matrix of undirected graphs). It is well known that the second smallest graph Laplacian eigenvalue, i.e., the algebraic connectivity of the graph, has the main role in the convergence time of various distributed algorithms. To speed up consensus algorithms, it has been shown that the optimal Laplacian-based consensus matrix is obtained with a step-size being equal to the inverse of the sum of the smallest and largest nonzero graph Laplacian eigenvalues [2]. One can do even more since all the spectrum of the Laplacian matrix can also be used for designing consensus matrices in order to achieve average consensus in a finite number of steps [3]. The Laplacian spectrum can also be used to assess the graph robustness through the Kirchhoff index [4], the connectedness of the graph, and can be utilized for locally checking the controllability and the observability of a networked system [5]. Therefore, estimation of the Laplacian eigenvalues is a problem of great interest for analyzing networked systems and synthesizing efficient distributed algorithms.

During the current decade, various studies have been carried out on decentralized estimation of Laplacian eigenvalues. In [6,7], the second smallest Laplacian eigenvalue was estimated by resorting to a decentralized power iteration method. In [8,9], Fast Fourier Transform (FFT)-based methods were suggested. The idea therein is to make the state of each agent oscillate only at frequencies corresponding to the eigenvalues of the graph Laplacian matrix associated with the network topology. Some particular protocols need to be set up in order to generate the data. Then the problem can be efficiently and independently solved by each agent in applying the FFT method. This approach inherits the limitations of the FFT algorithm (resolution between eigenvalues is strongly dependent on that of the FFT and the accuracy depends on the amount of the stored data). In addition, the node can only estimate the eigenvalues associated with the mode that is observable and some nodes may observe only a subset of the eigenvalues. In [10], data generated by a standard consensus protocol are used and a purely algebraic method using observability properties of the network is proposed. With this method, the eigenvalues of the network matrix can be recovered by solving a local eigenvalue decomposition on an appropriately constructed matrix of the observed data. However, this method is only applicable to networks having nodes with sufficient storage and computation capabilities. Precisely, several runs of the standard average consensus protocol, with different initial conditions, are needed.

In this paper, from a single run of a standard consensus protocol, we show how to estimate the Laplacian eigenvalues by solving the

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factorization of the averaging matrix $\mathbf{J}_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T$ in a distributed way. We generalize the contributions in [11,12] by considering an upper-bound for the number of distinct eigenvalues and formulating optimization problems with more adequate constraints. In contrast to [12], we consider that the data of the transient of the consensus protocol are stored for computing the exact average value, such as in [13], and then reused for graph Laplacian eigenvalues estimation.

The remainder of this paper is organized as follows: in Section 2, we first formulate the problem under study. Then, three algorithms are derived in Section 3 for solving constrained optimization problems. The performance of the proposed algorithms is evaluated in Section 4 by means of simulation results before concluding the paper.

2. Problem statement

Consider a network modeled with a connected undirected graph $G(V, E)$ where $V = \{1, 2, \ldots, N\}$ and $E \subset V \times V$ stand for the set of vertices (nodes) and that of edges respectively. We denote by $N_i = \{j \in V : (i, j) \in E\}$ the neighborhood of vertex i whereas $d_i = |N_i|$ stands for its degree. It is usual to capture interactions between nodes by means of the Laplacian matrix $\mathbf{L}$ whose entries are $l_{ii} = d_i$, $l_{ij} = -1$ if $(i, j) \in E$, and $l_{ij} = 0$ elsewhere. Its eigenvalues, $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$, are all real and nonnegative. They contain very significant information about the topology of the graph G [14] and can be used to assess the robustness of the graph. Let us define the state of the network as $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$. The purpose of average consensus is to reach an agreement after some message exchanges, the agreed value being the average of $\mathbf{x}(0)$. For this purpose, it is usual to resort to a linear iterations scheme. The most common time-invariant protocol is given by:

$$\mathbf{x}(t+1) = (\mathbf{I}_N - \alpha\mathbf{L})\,\mathbf{x}(t) = (\mathbf{I}_N - \alpha\mathbf{L})^t\mathbf{x}(0), \tag{1}$$

where α is selected such that $0 < \alpha < \frac{1}{d_{max}}$ to ensure asymptotic convergence [2], and $\mathbf{I}_N \in \Re^{N\times N}$ is an identity matrix. Defining $\mathbf{q}(t) = (q_1(t), \ldots, q_N(t))^T = \mathbf{L}^t\mathbf{x}(0)$, we can note that:

$$\mathbf{x}(t) = \sum_{\tau=0}^{t} f_{t,\tau}(\alpha)\mathbf{q}(\tau), \quad f_{t,\tau}(\alpha) = \binom{t}{\tau}(-\alpha)^\tau. \tag{2}$$

Conversely: $\mathbf{q}(t) = \frac{1}{f_{t,t}(\alpha)}\left(\mathbf{x}(t) - \sum_{\tau=0}^{t-1} f_{t,\tau}\mathbf{q}(\tau)\right)$, with $\mathbf{q}(0) = \mathbf{x}(0)$. It is also possible to consider a time-varying protocol, where the step-size α varies at each iteration:

$$\mathbf{x}(t+1) = (\mathbf{I}_N - \alpha_\tau\mathbf{L})\,\mathbf{x}(t) = \prod_{\tau=t}^{1}(\mathbf{I}_N - \alpha_\tau\mathbf{L})^t\mathbf{x}(0). \tag{3}$$

In such a case, we get:

$$\mathbf{x}(t) = \sum_{\tau=0}^{t} g_{t,\tau}(\alpha_1, \ldots, \alpha_t)\mathbf{q}(\tau), \tag{4}$$

where

$$g_{t,\tau}(\alpha_1, \ldots, \alpha_t) = (-1)^\tau \sum_{j_1=1}^{t}\sum_{j_2>j_1}^{t}\cdots\sum_{j_\tau>j_{\tau-1}}^{t}\prod_{q=1}^{\tau}\alpha_{j_q}, \tag{5}$$

$g_{t,0}(\alpha_1, \ldots, \alpha_t) = 1$, and we adopt the convention $\alpha_j = 0$ if $j > t$. One can note that α_t are the roots of the polynomial with coefficients $g_{t,\tau}(\alpha_1, \ldots, \alpha_t)$ which have an alternating sign pattern.

Using the time-invariant protocol (1), it is now well known that from consecutive measurements values of the state $x_i(t)$, node i can compute the exact average $\bar{x}$ of the initial condition [13]. These measurements can also be used to solve the averaging matrix factorization problem. The problem under study can therefore be formulated as follows:

Given intermediate observations $\mathbf{q}(t)$, $t = 0, \ldots, m$, *and the average value* $\bar{\mathbf{x}} = \mathbf{J}_N\mathbf{x}(0)$, *with* $\bar{x}(0)$ *an arbitrary initial condition of the consensus protocol, estimate in a distributed way the Laplacian eigenvalues of the graph representing the network.*

3. Laplacian eigenvalues estimation by means of a distributed averaging matrix factorization

In order to solve the problem formulated above, let us first state the following proposition:

Proposition 1. *Consider a connected graph with Laplacian matrix* $\mathbf{L}$ *and distinct nonzero eigenvalues* $\lambda_2, \ldots, \lambda_{D+1}$. *Let us define* $S_\mathbf{L} = \{1/\lambda_2, \ldots, 1/\lambda_{D+1}\}$ *and* $S_m = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. *Assume that the local values vector* $\mathbf{x}(0)$ *is not orthogonal to any eigenvector of* $\mathbf{L}$, *the cost function*

$$E(\boldsymbol{\alpha}) = \|\mathbf{x}(m) - \bar{\mathbf{x}}\|^2 = \left\|\prod_{t=m}^{1}(\mathbf{I}_N - \alpha_t\mathbf{L})\mathbf{x}(0) - \bar{\mathbf{x}}\right\|^2, \tag{6}$$

with $\boldsymbol{\alpha} = (\alpha_1 \cdots \alpha_m)^T$, $m \geq D$, *is minimal if and only if* $S_\mathbf{L} \subseteq S_m$.

Proof. Consider the eigenvalues decomposition of the Laplacian matrix: $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T$. Without loss of generality, we assume that the $(D+1)$ first entries of the diagonal matrix $\boldsymbol{\Lambda}$ are the $(D+1)$ distinct graph Laplacian eigenvalues. Since the graph is connected, we know that $\lambda_1 = 0$. Therefore, the corresponding eigenvector (first column of $\mathbf{U}$) is equal to $\frac{1}{\sqrt{N}}\mathbf{1}$. Let $\mathbf{e}_i$ be the i-th vector of the canonical basis of $\Re^N$. Then the averaging matrix $\mathbf{J}_N$ is equal to $\mathbf{U}diag(\mathbf{e}_1)\mathbf{U}^T$ where $diag(.)$ stands for a diagonal matrix with the vector in argument as main diagonal. We can therefore rewrite the cost function $E(\boldsymbol{\alpha})$ as:

$$\begin{aligned} E(\boldsymbol{\alpha}) &= \left\|\prod_{t=1}^{m}\left(\mathbf{I}_N - \alpha_t\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T\right)\mathbf{x}(0) - \mathbf{J}_N\mathbf{x}(0)\right\|^2 \\ &= \left\|\mathbf{U}\left(\prod_{t=1}^{m}(\mathbf{I}_N - \alpha_t\boldsymbol{\Lambda}) - diag(\mathbf{e}_1)\right)\mathbf{U}^T\mathbf{x}(0)\right\|^2 \\ &= \left\|\mathbf{U}diag\,(\boldsymbol{\varepsilon})\,\mathbf{U}^T\mathbf{x}(0)\right\|^2, \end{aligned}$$

with $\boldsymbol{\varepsilon} = (0, \prod_{t=1}^{m}(1-\alpha_t\lambda_2), \ldots, \prod_{t=1}^{m}(1-\alpha_t\lambda_N))^T$. Now, let us denote by w_i, $i = 1, \ldots, N$, the entries of $\mathbf{U}^T\mathbf{x}(0)$. Then: $E(\boldsymbol{\alpha}) = \sum_{i=2}^{N}\prod_{t=1}^{m}(1-\alpha_t\lambda_i)^2 w_i^2$. We can note that this function is a sum of square terms. Therefore, it vanishes if and only if each term $\prod_{t=1}^{m}(1-\alpha_t\lambda_i)^2 w_i^2$, $i = 2, \ldots, N$ is equal to zero. Since, $\mathbf{x}(0)$ is not orthogonal to any Laplacian eigenvector, then $w_i \neq 0, \forall i$. Therefore, $E(\boldsymbol{\alpha})$ vanishes if and only if $\prod_{t=1}^{m}(1-\alpha_t\lambda_i)^2 = 0$, $i = 2, \ldots, N$. Due to the multiplicities of the Laplacian eigenvalues, we have in fact D distinct equations $\prod_{t=1}^{m}(1-\alpha_t\lambda_i)^2 = 0$, $i = 2, \ldots, D+1$. These equations are fulfilled if and only if there are D coefficients α_t respectively equal to the inverse of the D distinct Laplacian eigenvalues. ■

Assuming that $\mathbf{x}(0)$ is randomly generated then almost surely it is not orthogonal to the eigenvectors of the Laplacian matrix. Therefore when the conditions of Proposition 1 are fulfilled, $\bar{\mathbf{x}} = \prod_{t=1}^{m}(\mathbf{I} - \alpha_t\mathbf{L})\mathbf{x}(0)$ or equivalently $\bar{\mathbf{x}} = \sum_{\tau=0}^{m} c_\tau\mathbf{q}(\tau) = \mathbf{Q}\mathbf{c}$, with $\mathbf{Q} = [\mathbf{q}(0)\ \mathbf{q}(1)\ \ldots\ \mathbf{q}(m)] \in \Re^{N\times(m+1)}$, $\mathbf{c} = [c_0\ c_1 \ldots\ c_m]^T \in \Re^{(m+1)}$, and $c_\tau = g_{m,\tau}(\alpha_1, \ldots, \alpha_m)$.

We can now state the following proposition which is a corollary of Proposition 1: