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Synchronization of pulse-coupled oscillators to a global pacemaker

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ABSTRACT

Pulse-coupled oscillators (PCOs) are limit cycle oscillators coupled by exchanging pulses at discrete time instants. Their importance in biology and engineering has motivated numerous studies aiming to understand the basic synchronization properties of a network of PCOs. In this work, we study synchronization of PCOs subject to a global pacemaker (or global cue) and local interactions between slave oscillators. We characterize solutions and give synchronization conditions using the phase response curve (PRC) as the design element, which is restricted to be of the delay type in the first half of the cycle, interval $(0, \pi)$, and of the advance type in the second half of the cycle, interval $(\pi, 2\pi)$. It is shown that global synchronization is feasible when using an advance-delay PRC if the influence of the global cue is strong enough. Numerical examples are provided to illustrate the analytical findings.

1. Introduction

Pulse-coupled oscillators (PCOs) are limit cycle oscillators that are coupled together to form a network by exchanging pulses at discrete time instants. A pulse has two effects on the network state: (1) it resets the phase at the originating oscillator, and (2) it induces a jump on the phase of the receiving oscillators. The magnitude of the impulsive jump induced is, in general, phase dependent and is given in the form of a phase response curve (PRC) Q [1]. Moreover, it is customary to include a coupling strength l to scale the effect of the PRC. In this setting, the value of l can be interpreted as the extra energy needed to synchronize the system, as is indeed the case when PCOs are realized using passive circuits, or as an extra gain present at the receiver side.

The dynamics of a network of PCOs, and thus its synchronization properties, are fully determined by the interaction topology (communication network) $\mathcal{R}$, the number of oscillators in the network N, the initial phases x_0, and the feedback strategy given by Q and l, i.e., the PRC and the coupling strength. Despite the simple formulation and behavior of an isolated firing oscillator, a network of PCOs is able to exhibit intricate collective dynamics. For this reason, PCOs have emerged as a powerful modeling and design tool in complex networked biological and engineering systems. Examples of biological systems that have been modeled using PCOs include cardiac pacemakers [2], crickets that chirp in unison [3], and rhythmic flashing of fireflies [4]. While one of the most important applications of PCOs in engineering is time synchronization in sensor networks [5–7]. Although PCOs form an impulsive network and can be interpreted as an event-triggered system, the results available for these classes of systems, such as [8,9], do not cover PCOs due to the oscillatory nature of the participating agents.

In a network of PCOs, the role of each agent, i.e., master or slave, also determines the resulting dynamics. In fact, in the achievement of synchronization the interplay between a global cue and local interactions between agents is an important feature [10]. For example, in the mammalian olfactory bulb, ensembles of neurons synchronize to discriminate odors by utilizing intercellular interplays among individual neurons while at the same time receiving a global driving odorant stimulus via the odorant receptors [11]. In engineering, the coordination of a network of unmanned ground vehicles is achieved by means of the interplay between individual vehicles and external coordination from the central resources [12].

In this work, we study the synchronization properties of a network of PCOs when there is a master node, or global cue, that can reach every other node and does not react to any incoming pulse. The global cue can be regarded either as an external input or as an internal leader acting inside the network. In particular, this

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work refines and extends the results in [10,13]. In [10], the weak coupling assumption [14] is used to transform the impulsive dynamics of a PCO network into an ordinary differential equation via averaging. Synchronization is proven to emerge for arbitrary initial conditions when an advance-delay PRC is used; however, the PRC is restricted to be a continuous function, which introduces a zero crossing point that precludes global synchronization. An important finding in [10] is that for a network of PCOs, global synchronization to a global cue is feasible only when the global cue can reach every other node. However, when the initial conditions are restricted to half of the circle, the global cue reaching a single node is a sufficient condition for synchronization. In [13], hybrid dynamical systems theory is used to allow the PRC to be a discontinuous mapping. However, the weak coupling assumption is also used, which limits the applicability in an artificial network of PCOs. Moreover, no guideline is given regarding the strength of the global coupling. In this paper, we remove the weak coupling assumption and prove that global synchronization is feasible when using a set-valued advance-delay PRC. Moreover, we provide a explicit bound for the global coupling that ensures global synchronization. We exploit the hybrid nature of pulse-coupled networks [15] to pose the synchronization problem as a set stabilization problem, which we solve using tools from hybrid systems theory.

1.1. Basic notation and definitions

In this work, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers, $\mathbb{R}_{<0(>0)}$ denotes the negative (positive) real numbers, $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes the Euclidean space of dimension n, and $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$ square matrices with real coefficients. For a countable set χ, we denote its cardinality as $|\chi|$; for two sets Λ_1 and Λ_2, we denote their difference as $\Lambda_1 \setminus \Lambda_2$. A set-valued mapping $\Phi : A \rightrightarrows B$ associates to the element $\alpha \in A$ the set $\Phi(\alpha) \subseteq B$; the graph of a set-valued mapping is the set: $\text{graph}(\Phi) := \{(\alpha, \beta) \in A \times B : \beta \in \Phi(\alpha)\}$. A set-valued mapping Φ is outer semi-continuous if and only if its graph is closed [16, Theorem 5.7(a)]. A function $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

1.2. Hybrid systems preliminaries

We follow the framework given in [17]. A hybrid system $\mathcal{H}$ consists of continuous-time dynamics (flows), discrete-time dynamics (jumps), and sets on which these dynamics apply:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in G(x), & x \in \mathcal{D} \end{cases} \tag{1}$$

where x is the state, the flow map F and the jump map G are set-valued mappings, $\mathcal{C} \subseteq \mathbb{R}^n$ is the flow set, and $\mathcal{D} \subseteq \mathbb{R}^n$ is the jump set, $(F, \mathcal{C}, G, \mathcal{D})$ is the data of $\mathcal{H}$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a hybrid time domain if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times j$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times j$, $[t_j, t_{j+1}) \times j$, or $[t_j, \infty) \times j$. A solution to $\mathcal{H}$ is a function $\phi : \text{dom}\,\phi \to \mathbb{R}^n$ where $\text{dom}\,\phi$ is a hybrid time domain and for each fixed j, $t \mapsto \phi(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \text{dom}\,\phi\}$. ϕ is called a hybrid arc, and is such that: for each $j \in \mathbb{N}$ for which I_j has nonempty interior $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I_j$, $\phi(t, j) \in \mathcal{C}$ for all $t \in [\min I_j, \sup I_j)$; for each $(t, j) \in \text{dom}\,\phi$ for which $(t, j+1) \in \text{dom}\,\phi$, $\phi(t, j+1) \in G(\phi(t, j))$, $\phi(t, j) \in \mathcal{D}$. A solution ϕ is nontrivial if its domain contains at least one point different from $(0, 0)$, is maximal if it cannot be extended, and is complete if its domain is unbounded.

1.3. Graph theory

Throughout this paper we use several concepts from algebraic graph theory [18]. Consider a network with $N \in \mathbb{Z}_{\geq 0}$ agents. The communication between agents is modeled by a weighted directed graph $\mathcal{R} = \{\mathcal{V}, \mathcal{E}_{\mathcal{R}}, \mathcal{A}_{\mathcal{R}}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ is the node set of the graph. $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of the graph, whose elements are such that $(i, k) \in \mathcal{E}_{\mathcal{R}}$ if and only if node k receives the pulse emitted by node i; we assume that the self edge $(i, i) \notin \mathcal{E}_{\mathcal{R}}$. $\mathcal{A}_{\mathcal{R}} = [a_{ik}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{R}$ with $a_{ik} \in \{0, l\}$, where $a_{ik} = l \in (0, 1]$ if and only if $(i, k) \in \mathcal{E}_{\mathcal{R}}$. For a node i, $\mathcal{N}^{i-} = \{k \in \mathcal{V} : (k, i) \in \mathcal{E}_{\mathcal{G}}\}$ denotes the in-neighbors of node i, i.e., the set of nodes whose pulses are received by i, and $\mathcal{N}^{i+} = \{k \in \mathcal{V} : (i, k) \in \mathcal{E}_{\mathcal{G}}\}$ denotes the out-neighbors of node i, i.e., the set of nodes that receive pulses emitted by i.

2. Model and problem formulation

Mirollo and Strogatz [19] presented the classical formulation of a network of PCOs. The network is formed by N oscillators, where each oscillator $i \in \{1, 2, \ldots, N\}$ follows

$$z_i = f(x_i), \tag{2}$$

where $f : [0, 1] \to [0, 1]$ is smooth, monotonically increasing, and concave down, i.e., $f'(x_i) > 0, f''(x_i) < 0$, and $x_i \in [0, 1]$ is a phase-like variable such that

$$\frac{\partial x_i}{\partial t} = \frac{1}{T} = \omega \tag{3}$$

and $x_i = 1$ ($x_i = 0$) when the oscillator is at the end (start) of the cycle, i.e., when $z_i = 1$ ($z_i = 0$). Therefore, $f(0) = 0$ and $f(1) = 1$ holds. The oscillators are assumed to interact by a simple form of pulse coupling: when an oscillator fires it increases the state of all the other oscillators by an amount ϵ, or forces them to fire, whichever is less. That is,

$$\begin{aligned} z_i(t) = 1 &\Rightarrow z_i(t^+) = 0 \\ &\Rightarrow z_j(t^+) = \min(1, z_j(t) + \epsilon), \quad \forall j \neq i. \end{aligned} \tag{4}$$

In the following, we reformulate the PCO model in the hybrid systems framework, which allows us to consider an arbitrary feedback mapping (in contrast to the constant ϵ) and include explicitly the structure of an underlying communication graph. The particular network structure considered is the one where an omnipresent master, or global cue, is part of the network, which we denote as the global cue or master node. In this setup, the network consists of a global cue and N slave oscillators aiming to synchronize their phases to the phase of the global cue. We assume that the slave oscillators interact on a given graph $\mathcal{R} = \{\mathcal{V}, \mathcal{E}_{\mathcal{R}}, \mathcal{A}_{\mathcal{R}}\}$, not necessarily connected. The phase of each slave oscillator evolves continuously following its natural frequency from 0 to 2π (in contrast to the range 0 to 1 as in [19]), and jumps impulsively upon receiving a pulse. The global cue is not affected by pulses, thus, its phase evolution is determined only by its natural frequency. Pulses are generated following an integrate-and-fire process, i.e., when its phase reaches 2π, the oscillator fires, i.e., emits a pulse, and resets its phase to 0. When an oscillator receives a pulse, it updates its phase according to the coupling strength and the PRC, which is defined in the framework of hybrid systems as follows:

Definition 1 (*Phase Response Curve*)**.** A phase response curve (PRC), or phase resetting curve [1,20], describes the change in the phase of an oscillator due to a pulse stimulus, as a function of the phase at which the pulse is received. A phase response curve $Q : [0, 2\pi] \rightrightarrows [0, \pi]$ is called an advance-only PRC. Similarly, a phase response curve $Q : [0, 2\pi] \rightrightarrows [-\pi, 0]$ is called a delay-only PRC. A phase response curve $Q : [0, 2\pi] \rightrightarrows [-\pi, \pi]$ is called an advance-delay PRC if there exist $\bar{q}_1 \in Q(q_1)$ and $\bar{q}_2 \in Q(q_2)$, with q_1 and q_2 in $[0, 2\pi]$, satisfying $\bar{q}_1 \in [-\pi, 0)$ and $\bar{q}_2 \in (0, \pi]$.