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Attraction and Lyapunov stability for control systems on vector bundles

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ABSTRACT

Let $\pi : E \to B$ be a finite-dimensional vector bundle whose base space is compact. In this paper, we study attraction and Lyapunov stability for control systems on E. We prove that, under certain conditions, the concepts of Conley attractor, uniform attractor, attractor, exponential attractor, asymptotically stable set and stable set are equivalent for the zero section of $\pi : E \to B$.

1. Introduction

The purpose of this paper is to study attraction and Lyapunov stability for control systems defined on total spaces of vector bundles.

Vector bundles appear naturally in the theory of control systems. In fact, for a given manifold M, it is well-known that the projection $\pi : TM \to M$ from the tangent bundle TM of M onto M is a vector bundle. Hence, if Σ is a control system on M, the linearization along the trajectories of Σ yields a bilinear control system on the tangent bundle TM.

The study of attraction and Lyapunov stability is classical in the theory of dynamical systems. In [1,2], Bhatia and Szegö present several concepts of attractors and Lyapunov stable sets for dynamical systems on metric spaces. Also, the concept of Conley attractor for dynamical systems on metric spaces was studied by Conley in [3,4]. The study of the asymptotic behavior of linear flows and control systems on vector bundles is also classical. It was treated by Ayala et al. in [5], Colonius and Kliemann in [6,7], Colonius et al. in [8], Grüne in [9], Salamon and Zehnder in [10], Selgrade in [11] and Souza in [12].

The study of the asymptotic behavior near the zero section of a vector bundle is usual in the theories of linear flows and control systems on vector bundles. It generalizes the study of the asymptotic behavior near the origin for a linear equation defined on an euclidean space. For instance, Fenichel's uniformity lemma (see [6, Lemma 5.2.7]) entails that the concepts of exponential attractor and attractor are equivalent for the zero section in the context of linear flows on vector bundles.

Recently, generalizations of the concepts of Conley attractor, chain recurrence and chain transitivity for semigroup actions on topological spaces were developed by Braga Barros and Souza in [13,14]. Several results of attractors and Lyapunov stable sets have also been studied in the context of semigroup actions on topological spaces by Braga Barros, Souza and Rocha in [15–17].

In [18], Braga Barros, Souza and Rocha present results on the relation among Conley attractors, attractors and Lyapunov stable sets for semigroup actions and control systems. In this paper, we apply the results obtained in [18] to relate these concepts for control systems on vector bundles. Since the dynamics of the class of control systems considered in this paper is given by means of the action of the semigroup of the system, we can apply here the results of the general theory of semigroup actions. We consider a control system defined on the total space of a finite-dimensional vector bundle whose base space is compact and study attraction and Lyapunov stability for the zero section of the vector bundle. The main result of this paper relates the concepts of Conley attractor, uniform attractor, attractor, asymptotically stable set and stable set for the zero section.

In Section 2, we recall the concepts and results that are used throughout this paper. We recall the definitions of limit sets and positive prolongational limit sets for control systems defined on manifolds and present some properties of these concepts. We also recall the concepts of attraction domains, attractors and Lyapunov stable sets for control systems and present properties of these sets.

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In the last section, we specialize to the study of attraction and Lyapunov stability for the zero section of a vector bundle. Here, the translation hypotheses (see Definition 2.1), especially the hypotheses called H_3 and H_4, are fundamental. These hypotheses have been used in the literature to study the asymptotic behavior of semigroup actions and control systems (for instance, see [13,14,19,15,17,18,20–22]). The hypothesis H_3 yields a relation among the concepts of stable weak attractor, attractor, uniform attractor, stable set and asymptotic stable set (see Proposition 2.2), while the hypothesis H_4 yields invariance for the ω-limit sets (see Proposition 2.1), which is fundamental in Lemma 3.1. We introduce the concepts of exponential attraction domain and exponential attractor of the zero section (see Definition 3.1). We show that, under certain conditions, the zero section is an attractor if and only if it is an exponential attractor (Corollary 3.1). This fact depends on Lemmas 3.1 and 3.2, which are generalizations of Lemmas 2.4 and 2.5 from [10], respectively. We also present a version of Fenichel's uniformity lemma for control systems on vector bundles (see Corollary 3.2). Finally, Theorem 3.1 provides an equivalence among attractors and Lyapunov stable sets, which is the main result of this paper.

2. Control systems

In this section, we recall some definitions and results on control systems that are used throughout this paper. We refer to [13,19,6,20–22] for the theory of control systems.

Let M be a finite-dimensional C^∞-manifold and let Σ be an *affine control system* on M given by

$$x'(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t))$$
$$u = (u_1, \ldots, u_m) \in \mathcal{U}_{pc}$$

where $X_0, \ldots, X_m$ are C^∞-complete vector fields in M and $\mathcal{U}_{pc} = \{u : \mathbb{R} \to U : u \text{ piecewise constant}\}$, with $U \subset \mathbb{R}^n$. We assume that, for each $u \in \mathcal{U}_{pc}$ and $x \in M$, the system Σ admits a unique solution $\varphi(t, x, u)$, $t \in \mathbb{R}$, with $\varphi(0, x, u) = x$. We use the notation $X(x, u(t)) = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x)$ and assume that $X_u = X(\cdot, u)$ is a C^∞-complete vector field in M, for every $u \in U$. For each $t \in \mathbb{R}$ and $u \in U$, we have the diffeomorphism $\varphi_t^u : M \to M$ defined by $\varphi_t^u(x) = \varphi(t, x, u)$. The *system semigroup* of the control system Σ is defined as

$$\mathcal{S}_\Sigma = \left\{\varphi_{t_n}^{u_n} \circ \cdots \circ \varphi_{t_1}^{u_1} : u_i \in U, t_i \geqslant 0, n \in \mathbb{N}\right\}.$$

It is easily seen that $\mathcal{S}_\Sigma$ acts on M as a semigroup of diffeomorphisms of M.

For an element $x \in M$ and a subset $A \subset \mathcal{S}_\Sigma$, we define

$$Ax = \left\{y \in M : \begin{array}{c}\text{there exists } \phi \in A\\ \text{such that } \phi(x) = y\end{array}\right\},$$
$$A^{-1}x = \left\{y \in M : \begin{array}{c}\text{there exists } \phi \in A\\ \text{such that } \phi(y) = x\end{array}\right\}.$$

The sets $\mathcal{S}_\Sigma x$ and $\mathcal{S}_\Sigma^{-1} x$ are respectively called the *positive* and the *negative orbit of Σ through $x \in M$*. Since the set of control functions of Σ is $\mathcal{U}_{pc}$, we have that

$$\mathcal{S}_\Sigma x = \left\{\begin{array}{c} y \in M : \text{ there exists } t \geqslant 0 \text{ and}\\ u \in \mathcal{U}_{pc} \text{ such that } \varphi(t, x, u) = y\end{array}\right\},$$
$$\mathcal{S}_\Sigma^{-1} x = \left\{\begin{array}{c} y \in M : \text{ there exist } t \geqslant 0 \text{ and}\\ u \in \mathcal{U}_{pc} \text{ such that } \varphi(t, y, u) = x\end{array}\right\}.$$

For subsets $X \subset M$ and $A \subset \mathcal{S}_\Sigma$, we define

$$AX = \bigcup_{x \in X} Ax \quad \text{and} \quad A^{-1}X = \bigcup_{x \in X} A^{-1}x.$$

We say that X is *positively* (respectively *negatively*) *invariant* for the system Σ if $\mathcal{S}_\Sigma X \subset X$ (respectively $\mathcal{S}_\Sigma^{-1} X \subset X$). Also, X is *invariant* for the system Σ if $\mathcal{S}_\Sigma X \subset X$ and $\mathcal{S}_\Sigma^{-1} X \subset X$. Finally, we say that X is *isolated invariant* for the system Σ if it is invariant and there exists a neighborhood N of X in M (which is said to be an *isolated neighborhood* of X) such that, for every $x \in N$, $\mathcal{S}_\Sigma x \cup \mathcal{S}_\Sigma^{-1} x \subset N$ implies $x \in X$.

For $t \geqslant 0$, we consider the set

$$(\mathcal{S}_\Sigma)_{\geqslant t} = \left\{\varphi_{t_n}^{u_n} \circ \cdots \circ \varphi_{t_1}^{u_1} : \begin{array}{c} u_i \in U, t_i \geqslant 0,\\ \sum_{i=1}^{n} t_i \geqslant t, n \in \mathbb{N}\end{array}\right\}.$$

The family

$$\mathcal{F}_{\text{ctr}} = \left\{(\mathcal{S}_\Sigma)_{\geqslant t} : t \geqslant 0\right\} \tag{1}$$

is a directed set when ordered by the reverse inclusion or, in other words, it is a time-depending filter basis on the subsets of $\mathcal{S}_\Sigma$ (that is, $\emptyset \notin \mathcal{F}_{\text{ctr}}$ and given $t, s \geqslant 0$, $(\mathcal{S}_\Sigma)_{\geqslant t+s} \subset (\mathcal{S}_\Sigma)_{\geqslant t} \cap (\mathcal{S}_\Sigma)_{\geqslant s}$).

Throughout this paper, we assume that the control range U of Σ is a compact and convex subset of $\mathbb{R}^n$. Thus, the closure of $\mathcal{U}_{pc}$ with respect to the weak* topology of $\mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^n)$, denoted by $\mathcal{U} = \text{cl}\left(\mathcal{U}_{pc}\right)$, is a compact Hausdorff space and the *solution map*

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \to M, \qquad (t, x, u) \mapsto \varphi(t, x, u) \tag{2}$$

is continuous, where $\varphi(t, x, u)$ is the unique solution of the system Σ with respect to the initial condition $x(0) = x$ and the function $u \in \mathcal{U}$ at the time t (see [18, Section 4] and [6, Sections 4.2 and 4.3]).

The following translation hypotheses on the family $\mathcal{F}_{\text{ctr}}$ were considered in [13,14,19,15,17,18,20–22].

Definition 2.1. We say that the system Σ **satisfies**

1. **the hypothesis** H_1 if for all $\phi \in \mathcal{S}_\Sigma$ and $t \geqslant 0$ there exists $s \geqslant 0$ such that $\phi(\mathcal{S}_\Sigma)_{\geqslant s} \subset (\mathcal{S}_\Sigma)_{\geqslant t}$.
2. **the hypothesis** H_2 if for all $\phi \in \mathcal{S}_\Sigma$ and $t \geqslant 0$ there exists $s \geqslant 0$ such that $(\mathcal{S}_\Sigma)_{\geqslant s}\phi \subset (\mathcal{S}_\Sigma)_{\geqslant t}$.
3. **the hypothesis** H_3 if for all $\phi \in \mathcal{S}_\Sigma$ and $t \geqslant 0$ there exists $s \geqslant 0$ such that $(\mathcal{S}_\Sigma)_{\geqslant s} \subset (\mathcal{S}_\Sigma)_{\geqslant t}\phi$.
4. **the hypothesis** H_4 if for all $\phi \in \mathcal{S}_\Sigma$ and $t \geqslant 0$ there exists $s \geqslant 0$ such that $(\mathcal{S}_\Sigma)_{\geqslant s} \subset \phi(\mathcal{S}_\Sigma)_{\geqslant t}$.

The system Σ satisfies the hypotheses H_1 and H_2 (see [13, Section 5] and [19, Section 4]). In the following, we present an example of a class of systems which satisfy the hypotheses H_3 and H_4.

Example 2.1. Let $M = \mathbb{R}^d$. Consider the bilinear control system Σ on M given by

$$x'(t) = \sum_{i=1}^{n} u_i(t) A_i(x(t)),$$

where $U = \{u \in \mathbb{R}^n : a \leqslant \|u\| \leqslant b\}$, with $a > 0$, and $A_1, \ldots, A_n \in \mathbb{R}^{d \times d}$ are pairwise commutative matrices. Then, the system Σ satisfies the hypotheses H_3 and H_4 (see [20, Example 2.4]).

For more examples of systems which satisfy the hypotheses H_3 and H_4, see [19–22].

The next concept of limit set for control systems was studied in [13, Section 5].

Definition 2.2. The ω-**limit set** of a subset $X \subset M$ for the system Σ is defined as

$$\omega(X) = \left\{\begin{array}{c} y \in M : \text{ there exist sequences}\\ t_n \to +\infty, (x_n) \text{ in } X \text{ and } (u_n) \text{ in } \mathcal{U}_{pc}\\ \text{such that } \varphi(t_n, x_n, u_n) \to y\end{array}\right\}.$$