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Closed-form solution for optimal convergence speed of multi-agent systems with discrete-time double-integrator dynamics for fixed weight ratios

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ABSTRACT

This paper considers the convergence speed of multi-agent systems with discrete-time double-integrator dynamics. The communication topology is assumed to be fixed and undirected. The speed of convergence of the associated average consensus protocol is analyzed, and the problem of maximizing the convergence speed over the free parameters in the consensus protocol is considered. A closed-form solution to this problem is proposed assuming that the ratios of weights of communication links are fixed. Furthermore it is shown that when the weight ratios are considered as decision variables, a global optimum of the convergence speed can be obtained by solving an LMI problem. Simulation results are provided that demonstrate the effectiveness of the proposed approach.

1. Introduction

The consensus problem for multi-agent systems has received considerable attention over the past ten years, because of its broad variety of applications in cooperative control, formation control, flocking, coverage control, task assignment and so on. Extensive surveys about the fields of applications are given e.g. in [1,2].

When multi-vehicle systems are required to reach consensus, the model of a single agent may be rather complex. Using a separation principle proposed in [3], which is based on an information flow filter approach, it is possible to design stabilizing local controllers for individual agents and a stabilizing information flow filter for the whole multi-vehicle system separately, see [4]. In practice, the most important filter types are single- and double-integrators, where typically the agents are required to agree on the position and/or the velocity.

Consensus theory for single-integrators has been intensively studied in the literature, see e.g. [5–7], where both continuous- and discrete-time consensus protocols are considered for networks with switching communication topology and time-delays. Similar attention has been paid to double-integrators in [8] for the continuous-time case and in [9–12] for the discrete-time case. Since in practice communication between the agents involves sampled rather than continuous-time data, in this paper we focus on discrete-time systems.

Much effort has been spent on the analysis of the convergence speed of discrete-time single-integrators, see [13,14]. In [13] the convergence speed for a fixed and undirected graph is optimized by solving an LMI problem, whereas in [14] the convergence speed of a decentralized convergence strategy is analyzed and bounds are given. In contrast, not much work has been reported on double-integrator systems. In [15] the convergence properties of continuous-time double-integrator dynamics are studied for fixed and undirected graphs. Motivated by this work, the present paper studies the convergence speed of discrete-time double-integrator dynamics with fixed undirected communication graphs. For a given communication topology and fixed sampling time the convergence speed for the discrete-time double-integrator dynamics is optimized over three parameters: a gain α, the scaling of the weighting matrix W and the ratio of the weights in W. This optimization in three steps is similar to that in [16]. Although a multi-rate process is considered there, for the optimization the problem there is simplified by assuming identical periods for all agents. Thus with some redefinition of the optimization parameters the problems considered here and in [16] are identical. But while in [16], the second step is solved by bisection and the third one by a line search, here closed-form solutions for both step are

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provided. The analytic solution only depends on the largest and smallest eigenvalues of the Laplacian and thus the complexity does not scale with the network size. For the last parameter an LMI problem has to be solved that scales with the network size. For very large networks this step can be left out, numerical examples show that the main improvement with respect to the convergence speed is gained by the analytic solutions. The detailed insight into the dependence of the poles of the consensus process on the parameters, makes it possible to extend the results to other characteristics as the damping, which is also discussed here.

The remainder of this paper is organized as follows. In Section 2 some concepts from graph theory are reviewed, the discrete-time consensus protocol considered here is presented together with its stability bounds, and the problem statement is given. Section 3 contains the main contribution of this work: the influence of the free parameters of the consensus protocol on the convergence speed is analyzed, and optimal solutions for fastest convergence are proposed. In Section 4 it is shown how the results of Section 3 can be extended to satisfy requirements other than speed such as damping. Simulation results illustrate the approach in Section 5, and conclusions are drawn in Section 6.

2. Preliminaries

2.1. Graph theory

Consider a multi-agent system with n agents and m communication links, described by the undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$ with the node set $\mathcal{N} = \{1, \dots, n\}$, which represents the agents, and the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ describing the communication topology. If there is an edge $\{ij\} \in \mathcal{E}$ then agent i receives information from agent j and, since undirected graphs are considered, vice versa. The weighting matrix $W \in \mathcal{R}^{m\times m}$ is a diagonal matrix with $w_{ll} = w_{\{ij\}}$ being the weight on the lth edge $\{ij\}$. The weighted Laplacian for undirected graphs can be calculated as $L = DWD^T$, where $D \in \mathcal{R}^{n\times m}$ is the incidence matrix: if the lth edge is $\{ij\}$ then $d_{il} = 1$, $d_{jl} = -1$ and $d_{kl} = 0$ for $k \neq i, j$. The resulting Laplacian L does not depend on the choice of the edge direction $\{ij\}$ or $\{ji\}$. Due to construction 0 is an eigenvalue of L corresponding to the eigenvector $\mathbf{1}$ (a vector with all elements equal to 1) and the eigenvalues of L can be ordered as $\lambda_1 = 0 \le \lambda_2 \le \dots \le \lambda_n$ with $\lambda_i \in \mathcal{R}\ \forall i = 1, \dots, n$, because of the symmetry of L. If and only if the graph is connected, we have $\lambda_2 > 0$.

2.2. Consensus protocol

The discrete-time double-integrator dynamics of the ith agent is given by

$$\xi_i[k+1] = \xi_i[k] + \tau\zeta_i[k], \qquad \zeta_i[k+1] = \zeta_i[k] + \tau u_i[k],$$

where $\tau > 0$ can be seen as sampling interval. With the distributed discrete-time consensus protocol

$$u_i[k] = -\sum_{j=1}^{n} w_{\{ij\}} \left[(\xi_i[k] - \xi_j[k]) + \alpha(\zeta_i[k] - \zeta_j[k])\right]$$

proposed in [10] with $\alpha > 0$, this leads to

$$\begin{bmatrix}\xi[k+1]\\ \zeta[k+1]\end{bmatrix} = \begin{bmatrix} I_n & \tau I_n \\ -\tau L & I_n - \tau\alpha L\end{bmatrix}\begin{bmatrix}\xi[k]\\ \zeta[k]\end{bmatrix} = \Psi\begin{bmatrix}\boldsymbol{\xi}[k]\\ \boldsymbol{\zeta}[k]\end{bmatrix}. \quad (1)$$

Here $\xi[k] = [\xi_1[k], \dots, \xi_n[k]]^T$ and $\zeta[k]$ is defined respectively. It is assumed here for ease of notation that ξ_i and ζ_i are one-dimensional signals. The results can easily be extended to vector-valued signals by using the Kronecker product.

2.3. Convergence analysis of consensus protocol

We start the discussion of convergence by calculating the characteristic polynomial of Ψ in (1) as

$$p(s) = \det(sI - \Psi) = \prod_{i=1}^{n}\left(s^2 + s(-2 + \tau\alpha\lambda_i) + 1 + \tau^2\lambda_i - \tau\alpha\lambda_i\right) \quad (2)$$

where λ_i denotes the ith eigenvalue of L. It is obvious that for every λ_i there are two corresponding eigenvalues of Ψ, denoted by ψ_{2i-1} and ψ_{2i}, which can be calculated as

$$\psi_{2i-1,2i} = 1 - \frac{\tau}{2}\alpha\lambda_i \pm \frac{\tau}{2}\sqrt{\alpha^2\lambda_i^2 - 4\lambda_i}. \quad (3)$$

Thus $\lambda_1 = 0$ leads to $\psi_1 = \psi_2 = 1$. Average consensus is reached asymptotically if and only if $|\psi_j| < 1$ for $j = 3, \dots, 2n$.

Lemma 1. *The consensus protocol (1) reaches average consensus, i.e.* $|\psi_j| < 1$ *for* $j = 3, \dots, 2n$, *iff*

$$\alpha_{\mathrm{lb}} = \tau < \alpha < \frac{2}{\lambda_n\tau} + \frac{\tau}{2} = \alpha_{\mathrm{ub}}. \quad (4)$$

Proof. We apply the bilinear transformation $s = \frac{t+1}{t-1}$ to (2) as proposed in [9] leading to

$$p(t) = \prod_{i=1}^{n}\left(t^2(\tau^2\lambda_i) + t(2\tau\alpha\lambda_i - 2\tau^2\lambda_i) + (4 + \tau^2\lambda_i - 2\tau\alpha\lambda_i)\right).$$

This transformation maps roots in the open unit disk into the open left half plane. Thus $|\psi_j| < 1$ for $j = 3, \dots, 2n$ is equivalent to all roots of $p(t)$ being in the open left half plane. Using the Routh–Hurwitz stability criterion this is the case if and only if all coefficients are positive. The first coefficient $\tau^2\lambda_i$ is always positive, the condition resulting from the second one leads to the lower bound and that for the third to the upper one. □

Substituting the lower bound α_{lb} for α in (3) leads to $|\psi_j| = 1$ for $j = 1, \dots, 2n$. If the upper bound α_{ub} is substituted in (3) then $|\psi_{2n}(\alpha_{\mathrm{ub}})| = 1$ and $|\psi_j(\alpha_{\mathrm{ub}})| < 1$ for $j = 3, \dots, 2n-1$ if $\lambda_n < \frac{1}{\tau^2}$; this will be proved in Section 3.1.

2.4. Problem statement

In the following the sampling interval τ as well as the communication topology, represented by the incidence matrix D, are assumed to be given. The task considered here is to determine α and choose the edge weights W such that (1) converges as fast as possible. Since $\psi_1 = \psi_2 = 1$ and $|\psi_j| < 1$ for $j = 3, \dots, 2n$, the eigenvalue of Ψ with third largest absolute value determines the speed of the convergence process and is therefore used as a measure of convergence r here. Thus the task can be expressed in the form of a min–max optimization problem

$$r = \min_{\alpha, W}\max_{j}|\psi_j|, \quad j = 3, \dots, 2n. \quad (5)$$

It is easy to transform that problem into a matrix inequality, but this would be of size $2n \times 2n$ and bilinear in α and L. Here instead an optimal analytic solution for α and the scaling of W is given, whose complexity does not scale with the network size. An optimization of the weight ratios with the help of an LMI of size $n \times n$ is derived, that leads together with the proposed analytic solutions to the global optimal solution.

3. Consensus speed

In this section the influence of α and W on the convergence speed and closed-form solutions to (5) are discussed. This discussion is led in three stages, which are similar to those in [16], but