Systems & Control Letters 90 (2016) 1–6

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

The maximum principle for stochastic differential systems with general cost functional

Shuzhen Yang

Institution of Financial Studies, Shandong University, Jinan, Shandong 250100, PR China

A R T I C L E I N F O

Article history:
Received 9 May 2015
Received in revised form
4 January 2016
Accepted 5 January 2016
Available online 28 January 2016

Keywords:
Stochastic differential equations
Stochastic maximum principle
Hamilton systems

A B S T R A C T

In this paper, under the framework of Fréchet derivatives, we study a stochastic optimal control problem driven by a stochastic differential equation with general cost functional. By constructing a series of first-order and second-order adjoint equations, we establish the stochastic maximum principle and get the related Hamilton systems.

1. Introduction

It is well known that dynamic programming with related HJB equations and maximum principle are powerful approach to solving optimal control problems (see [1–6]). The HJB equations derived for stochastic delay systems (see [7–9]) and dynamic programming principle for functional systems (see [10]).

In the classical stochastic optimal control problem case, one studied the optimal control problem which is described by the following stochastic differential equation

$$X(s)=\int_0^s b(X(t),u(t))dt+\int_0^s \sigma(X(t),u(t))dW(t)$$

with the cost functional

$$J(u(\cdot))=E\left[\int_0^T f(X(t),u(t))dt+\Psi(X(T))\right],$$

for more details see Peng [1] who gave the general maximum principle for the above model. Furthermore, we may pay attention to the states $X(t_1),X(t_2),\ldots,X(t_n)$ with $0<t_1<t_2\ldots<t_n=T$ when we consider the cost functional, i.e.

$$J(u(\cdot))=E\left[\int_0^T f(X(t),u(t))dt+\sum_{i=1}^n\Psi(X(t_i))\right].$$

In many real world applications, the systems can be modeled by differential systems whose evolutions depend on the states. Therefore, in this study, we consider the following stochastic differential equation

$$X(s)=\int_0^s b(X(t),u(t))dt+\int_0^s \sigma(X(t),u(t))dW(t)$$

with a general cost functional

$$J(u(\cdot))=E\left[\int_0^T f(X(t),u(t))dt+\Psi(X_T)\right],$$

where $X_T:=X(s)_{0\le s\le T}$, means the path of X from 0 to T.

Thus, we study the stochastic maximum principle for the above stochastic differential systems. The main difficult is that the cost functional has the part $\Psi(X_T)$, which is the functional of $X(s)_{0\le s\le T}$. By the Riesz representation theorem, the Fréchet derivatives $D\Psi(X_T)$ and $D^2\Psi(X_T)$ can be described by a finite measure μ and a bilinear form β. Furthermore, we need to transform the bilinear form β into another representation. Then, by decomposing the measures μ and β as continuous parts and jump parts, we construct a series of first-order and second-order adjoint equations and get the stochastic maximum principle.

The paper is organized as follows: In Section 2, we present the stochastic optimal control problem. The proof of maximum principle theorem is given in Section 3.

2. The optimal control problem

Let W be a 1-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega,\mathcal{F},P;\{\mathcal{F}(t)\}_{t\ge0})$, where $\{\mathcal{F}(t)\}_{t\ge0}$ is the P-augmentation of the natural filtration generated by the Brownian motion W.

E-mail address: yangsz@sdu.edu.cn.

http://dx.doi.org/10.1016/j.sysconle.2016.01.001

Let $T > 0$ be given, consider the following controlled stochastic differential equation:

$$dX(s) = b(X(s), u(s))ds + \sigma(X(s), u(s))dW(s), \quad s \in (0, T], \quad (2.1)$$

with the initial condition $X(0) = x$, where $u(\cdot) = \{u(s), s \in [0, T]\}$ is a control process taking value in a compact set U of $\mathbb{R}$ and b, σ are given deterministic functions.

The cost functional is as follows:

$$J(u(\cdot)) = E\left[\int_0^T f(X(t), u(t))dt + \Psi(X_T)\right], \quad (2.2)$$

where $X_T := X(s)_{0\leq s\leq T}$, means the path of X from 0 to T and

$$\begin{aligned} b &: \mathbb{R} \times U \to \mathbb{R},\\ \sigma &: \mathbb{R} \times U \to \mathbb{R},\\ f &: \mathbb{R} \times U \to \mathbb{R},\\ \Psi &: C[0, T] \to \mathbb{R}, \end{aligned}$$

where $C[0, T]$ is the set of continuous functions on $[0, T]$.

Let $h := b, \sigma, f$, we assume h uniformly continuous and satisfy the following linear growth and Lipschitz conditions.

Assumption 2.1. Suppose there exists a constant $c > 0$ such that

$$|h(x^1, u) - h(x^2, u)| \leq c|x^1 - x^2|,$$

$$\forall (x^1, u), (x^2, u) \in \mathbb{R} \times U.$$

Assumption 2.2. Suppose there exists a constant $c > 0$ such that

$$|h(x, u)| \leq c(1 + |x|), \quad \forall (x, u) \in \mathbb{R} \times U.$$

Let Ψ be uniformly continuous real-valued functionals on $C[0, T]$, respectively.

Assumption 2.3. Suppose there exists a constant $c > 0$ such that

$$|\Psi(\Phi_T^1) - \Psi(\Phi_T^2)| \leq c\|\Phi_T^1 - \Phi_T^2\|,$$

$\forall (\Phi_T^1, \Phi_T^2) \in C[0, T] \times C[0, T]$ and $\|\cdot\|$ is the maximum norm on $C[0, T]$.

Assumption 2.4. Let h be differentiable at x, Ψ be Fréchet differentiable and there exists a constant $c > 0$ such that

$$\begin{aligned} &|\partial_x h(x^1(t), u^1) - \partial_x h(x^2(t), u^2)| \leq c(|x^1(t) - x^2(t)| + |u^1 - u^2|),\\ &|(D\Psi(\Phi_T^1) - D\Psi(\Phi_T^2))(1_{[0,T]})| \leq c\|\Phi_T^1 - \Phi_T^2\|,\\ &|(D^2\Psi(\Phi_T^1) - D^2\Psi(\Phi_T^2))(1_{[0,T]}, 1_{[0,T]})| \leq c\|\Phi_T^1 - \Phi_T^2\|, \end{aligned}$$

$$\forall (t, x_T^1, x_T^2, u^1, u^2) \in [0, T] \times C[0, T] \times C[0, T] \times U \times U.$$

Let $\mathcal{U}[0, T] = \{u(\cdot) \in L^2_{\mathcal{F}}(0, T; U)\}$. Suppose Assumptions 2.1 and 2.2 hold, then there exists a unique solution X for Eq. (2.1) (see [11]).

Minimize (2.2) over $\mathcal{U}[0, T]$. Any $\bar{u}(\cdot) \in U[0, T]$ satisfying

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)) \quad (2.3)$$

is called an optimal control. The corresponding state trajectory $(\bar{u}(\cdot), \bar{X}(\cdot))$ is called an optimal state trajectory and optimal pair.

Firstly, we give the well known pontryagin's stochastic maximum principle, in which we will show the second order necessary conditions for optimal pairs.

Under the framework of Fréchet derivatives, for $\Psi(\bar{X}_T)$, by the Riesz representation theorem, there is a unique finite Borel measure μ on $[0, T]$ such that $\forall \eta_T \in C[0, T]$

$$D\Psi(\bar{X}_T)(\eta_T) = \int_{[0,T]} \eta(s)d\mu(s). \quad (2.4)$$

Because μ is a finite measure on $[0, T]$, the measure of μ at most countable points are positive. Denote as $\{u(\{t_i\})\}_{i=1}^{+\infty}$, and we suppose that $0 = \cdots < t_2 < t_1 = T$.

And there is bilinear form $\beta : C[0, T] \times C[0, T] \to \mathbb{R}$ such that

$$D^2\Psi(\bar{X}_T)(\eta_T, \eta_T) = \int_{[0,T]\times[0,T]} \eta(t)\eta(s)d\beta(t, s). \quad (2.5)$$

By the symmetry of second derivatives, we have $\beta(s, t) = \beta(t, s), (t, s) \in [0, T] \times [0, T]$, $\beta(s, t)$ is a finite measure on $[0, T] \times [0, T]$, so there exist at most countable points' measure are positive, denote the jump points as $(s_i, k_j)_{i,j=1}^{\infty}$ and suppose $0 = \cdots s_2 < s_1 = T; 0 = \cdots k_2 < k_1 = T$. Note that $\beta(\{s\}, \{t\}) = \beta(\{t\}, \{s\}), (t, s) \in [0, T] \times [0, T]$, and

$$\beta(\{s_i\}, \{s_i\}) + \beta(\{k_j\}, \{k_j\}) = \beta(\{s_i\}, \{k_j\}) + \beta(\{k_j\}, \{s_i\}) > 0. \quad (2.6)$$

For convenience, we denote the sets $(s_i, s_i)_{i=1}^{\infty}$ and $(k_j, k_j)_{j=1}^{\infty}$ as $(l_i, l_i)_{i=1}^{\infty}$, then the set $(l_i, l_i)_{i=1}^{\infty}$ contains all the jumps points of β. We also denote the sets $\{t_i\}_{i=1}^{\infty}$ and $\{l_i\}_{i=1}^{\infty}$ as $\{h_i\}_{i=1}^{\infty}$ with $0 = \cdots h_2 < h_1 = T$.

Based on the above analysis, we introduce the first-order and second-order adjoint equations as follows:

The first-order adjoint equations are

$$\begin{aligned} -dp(t) &= \{\partial_x b(\bar{X}(t), \bar{u}(t))p(t) + \partial_x\sigma(\bar{X}(t), \bar{u}(t))q(t)\\ &\quad - \mu'(t) - \partial_x f(\bar{X}(t), \bar{u}(t))\}dt - q(t)dW(t), \quad t \in (h_{i+1}, h_i), \quad (2.7)\\ -p(t_i) &= \mu(\{h_i\}) - p(h_i^+), \quad i = 1, 2, 3, \ldots \end{aligned}$$

where h_i^+ is the right limit of h_i, $\mu'(t)$ is the derivative of $\mu(t)$, and $p(h_1^+) = 0$.

Denote that

$$H(x, u, p, q) = b(x, u)p + \sigma(x, u)q - f(x, u),$$
$$(x, u, p, q) \in \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}.$$

The second-order adjoint equations are

$$\begin{aligned} -dP(t) &= \{2\partial_x b(\bar{X}(t), \bar{u}(t))P(t)\\ &\quad + \partial_x\sigma(\bar{X}(t), \bar{u}(t))P(t)\partial_x\sigma(\bar{X}(t), \bar{u}(t))\\ &\quad - \gamma'(t) + 2\partial_x\sigma(\bar{X}(t), \bar{u}(t))Q(t)\\ &\quad + \partial_{xx}H(\bar{X}(t), \bar{u}(t), p(t), q(t))\}dt\\ &\quad - Q(t)dW(t), \quad t \in (h_{i+1}, h_i), \quad (2.8)\\ -P(h_i) &= \beta(\{h_i\}, \{h_i\}) - P(h_i^+), \quad i = 1, 2, 3, \ldots \end{aligned}$$

where

$$\gamma'(t) = \int_{h_{i+1}}^{h_i} d\beta(s, t)$$

and $P(h_1^+) = 0, t \in (h_{i+1}, h_i), \ i = 1, 2, 3, \ldots$

The main result is the following theorem:

Theorem 2.5. *Let Assumptions 2.1–2.4 hold, and $(\bar{u}(\cdot), \bar{X}(\cdot))$ be an optimal pair of (2.3). Then there exist $(p(\cdot), q(\cdot))$ and $(P(\cdot), Q(\cdot))$ satisfying the series of first-order adjoint equations (2.7) and second-order adjoint equations (2.8) and respectively such that*

$$\begin{aligned} &(H(\bar{X}(t), \bar{u}(t), p(t), q(t)) - H(\bar{X}(t), u, p(t), q(t)))\\ &\quad - \frac{1}{2}(\sigma(\bar{X}(t), \bar{u}(t)) - \sigma(\bar{X}(t), u))^2P(t) \geq 0, \quad (2.9) \end{aligned}$$

for any $u \in U$ and $t \in (h_{i+1}, h_i), \ i = 1, 2, 3 \ldots$.

Before to prove the maximum principle, we show an example to verify Theorem 2.5.