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On the necessity of the invariance conditions for reach control on polytopes[☆]

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ABSTRACT

We study the Reach Control Problem (RCP) to make the solutions of an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. So-called *invariance conditions* are used to prevent solutions from leaving the polytope through facets which are not designated as the exit facet. These conditions are known to be necessary for solvability of the RCP on polytopes by continuous state feedback. We study whether the invariance conditions are also necessary for solvability of the RCP on polytopes by open-loop controls. We show by way of a counterexample that surprisingly the answer is negative. We identify a suitable class of polytopes for which the invariance conditions remain necessary conditions.

1. Introduction

We study the Reach Control Problem (RCP) for affine systems on polytopes. The problem is to design a state feedback to force closed-loop solutions starting anywhere in a polytopic state space $\mathcal{P}$ to leave the polytope from a prescribed exit facet of the polytope in finite time [1–3]. The RCP is a fundamental reachability problem for piecewise affine hybrid systems [4,5]. The problem has been developed in [1,5–10] for simplices and [1,11,2,3] for polytopes. In these papers the *invariance conditions* are used to prevent solutions from leaving the polytope from facets which are not designated as the exit facet. They were shown to be necessary conditions for solvability of the RCP on polytopes by continuous state feedback in [1] and to be necessary conditions for solvability of the RCP on simplices by open-loop controls in [9]. In this note we show that the invariance conditions are not necessary conditions for solvability of the RCP on polytopes by open-loop controls. We prove that for a special class of polytopes the invariance conditions remain necessary conditions using open-loop controls. The result extends both [1,9], and opens the door for determining the largest feedback class needed to solve the RCP on polytopes.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The notation $\mathcal{K}_1 \setminus \mathcal{K}_2$ denotes elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$. The notation $\mathcal{B}$ denotes the open ball of radius 1 centered at the origin. For two vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. Let $T_{\mathcal{P}}(x)$ denote the Bouligand tangent cone to set $\mathcal{P} \subset \mathbb{R}^n$ at point x [12]. A set-valued map $\mathcal{Y} : \mathbb{R}^n \to 2^{\mathbb{R}^q}$ is said to be *upper semicontinuous* at $x \in \mathbb{R}^n$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - x'\| < \delta \Rightarrow \mathcal{Y}(x') \subset \mathcal{Y}(x) + \epsilon\mathcal{B}$.

2. Reach control problem

Consider an n-dimensional polytope $\mathcal{P} := \mathrm{co}\{v_1, \ldots, v_p\}$ with vertex set $V := \{v_1, \ldots, v_p\}$. An *edge* of $\mathcal{P}$ is a 1-dimensional face of $\mathcal{P}$, and a *facet* of $\mathcal{P}$ is an $(n-1)$-dimensional face of $\mathcal{P}$. Let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$ denote the facets of $\mathcal{P}$. The facet $\mathcal{F}_0$ is called the *exit facet* and facets $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are called the *restricted facets*. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J := \{1, \ldots, r\}$, and $J(x) := \{j \in J \mid x \in \mathcal{F}_j\}$. That is, $J(x)$ is the set of indices of the restricted facets that contain x. For each $x \in \mathcal{P}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ y \in \mathbb{R}^n \mid h_j \cdot y \le 0,\ j \in J(x) \right\}. \quad (1)$$

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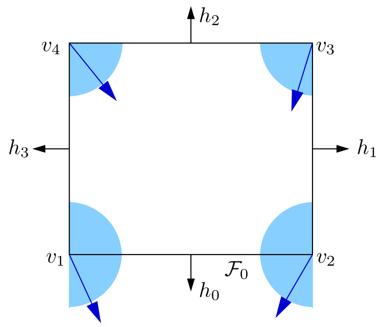

Fig. 1. The convex cones $\mathcal{C}(v_i)$ in a two-dimensional polytope.

Note that the index 0 never appears in $J(x)$ since $\mathcal{F}_0$ is the exit facet. For any $x \in \mathcal{P} \setminus \mathcal{F}_0$, $\mathcal{C}(x) = T_{\mathcal{P}}(x)$, the Bouligand tangent cone to $\mathcal{P}$ at x. Instead, at $x \in \mathcal{F}_0$, $\mathcal{C}(x)$ and $T_{\mathcal{P}}(x)$ are different since $\mathcal{C}(x)$ includes directions pointing out of $\mathcal{P}$. See Fig. 1. We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a, \quad x \in \mathcal{P}, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}B$, the image of B. Also define $\mathcal{O} := \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$ and $\mathcal{O}_{\mathcal{P}} := \mathcal{P} \cap \mathcal{O}$, the set of all possible equilibrium points of (2) in $\mathcal{P}$. We say that a function $\mu : [0, \infty) \to \mathbb{R}^m$ is an *open-loop control* for (2) if it is bounded on any compact time interval and it is measurable. These standard conditions ensure the existence and uniqueness of solutions of (2). Let $\phi_u(t, x_0)$ denote the trajectory of (2) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 2.1 (*Reach Control Problem (RCP)*)**.** Consider system (2) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that for every $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

The RCP says that the solutions of (2) starting from initial conditions in $\mathcal{P}$ reach and exit $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. In the sequel we use the shorthand notation $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ to denote that the RCP is solvable by some control.

Definition 2.1. We say the *invariance conditions are solvable* if for each $v \in V$, there exists $u \in \mathbb{R}^m$ such that

$$Av + Bu + a \in \mathcal{C}(v). \tag{3}$$

Eq. (3) is referred to as the *invariance conditions* either for a specific vertex, or collecting all conditions for all vertices, for a polytope.

3. Counterexample

The invariance conditions (3) are known to be necessary for solvability of the RCP on polytopes by continuous state feedback [1] and on simplices by open-loop controls [9]. We show by way of a counterexample that for general polytopes and open-loop controls, the invariance conditions are, however, no longer necessary.

We consider the polytope $\mathcal{P} = \text{co}\{v_1, \ldots, v_5\} \subset \mathbb{R}^3$ shown in Fig. 2 with vertices $v_1 = (\frac{1}{2}, 1, 1)$, $v_2 = (0, 1, 0)$, $v_3 = (1, 1, 0)$, $v_4 = (1, 0, 0)$, and $v_5 = (0, 0, 0)$. The exit facet is $\mathcal{F}_0 = \text{co}\{v_1, v_2, v_3\}$, depicted as a hatched region in the figure. The restricted facets are $\mathcal{F}_1 = \text{co}\{v_1, v_2, v_5\}$, $\mathcal{F}_2 = \text{co}\{v_1, v_3, v_4\}$, and $\mathcal{F}_3 = \text{co}\{v_1, v_4, v_5\}$. Also, $h_1 = (\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, $h_2 = (\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, and $h_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Notice that the two hyperplanes containing the facet $\mathcal{F}_1 = \text{co}\{v_1, v_2, v_5\}$ and the facet $\mathcal{F}_2 = \text{co}\{v_1, v_3, v_4\}$ intersect at the line through v_1 and $v_1\prime$, and this line is parallel to the horizontal hyperplane containing $\text{co}\{v_2, v_3, v_4, v_5\}$.

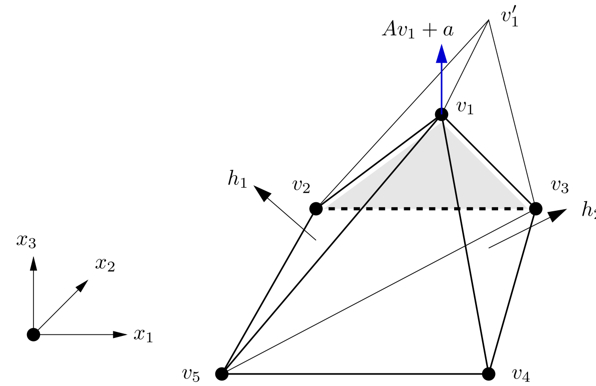

Fig. 2. The invariance conditions are not solvable but the RCP is solvable by open-loop controls.

The affine control system is

$$\dot{x} = Ax + Bu + a = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ \frac{1}{10} & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

The set of possible equilibria of this system is $\mathcal{O} = \{x \in \mathbb{R}^3 \mid \frac{1}{10}x_1 + x_3 = -1\}$. It can be verified that $\mathcal{P} \cap \mathcal{O} = \emptyset$. This means that for all $x \in \mathcal{P}$ and all $u \in \mathbb{R}^2$, $Ax + Bu + a \neq 0$.

We show that the invariance conditions of $\mathcal{P}$ at v_1 are not solvable. We observe that $v_1 \in \mathcal{F}_0 \cap \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$, so $J(v_1) = \{1, 2, 3\}$. The condition (3) at v_1 states that there must exist $u_1 \in \mathbb{R}^2$ such that

$$\begin{aligned} y_1 &:= Av_1 + Bu_1 + a \in \mathcal{C}(v_1) \\ &= \left\{y \in \mathbb{R}^3 \mid h_j \cdot y \leq 0,\ j \in \{1, 2, 3\}\right\}. \end{aligned}$$

Letting $u_1 = (u_{11}, u_{12})$ and substituting numerical values, we have $y_1 = (u_{11}, u_{12}, 2 + \frac{1}{20})$. The condition $h_1 \cdot y_1 \leq 0$ becomes $u_{11} \geq 1.025$, and the condition $h_2 \cdot y_1 \leq 0$ becomes $u_{11} \leq -1.025$. Clearly these two conditions cannot be solved simultaneously for u_{11}, so there does not exist $u_1 \in \mathbb{R}^2$ so that (3) holds at v_1. We conclude the invariance conditions of $\mathcal{P}$ are not solvable at v_1.

Second, we show there exist open-loop controls solving the RCP on $\mathcal{P}$. In fact, we construct a piecewise affine feedback that solves the problem. First we triangulate $\mathcal{P}$ using the triangulation $\mathbb{T} = \{\mathcal{S}_1, \mathcal{S}_2\}$, where $\mathcal{S}_1 = \text{co}\{v_1, v_2, v_3, v_5\}$ and $\mathcal{S}_2 = \text{co}\{v_1, v_3, v_4, v_5\}$ are two simplices, as shown in Fig. 2. Second, we split the control objective as $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback, where $\mathcal{F} := \mathcal{S}_1 \cap \mathcal{S}_2 = \text{co}\{v_1, v_3, v_5\}$, and $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback.

It is well-known that the RCP is solvable by affine feedback on simplices if and only if the invariance conditions of the simplex are solvable and the unique affine feedback constructed from one choice of solution of the invariance conditions does not admit a closed-loop equilibrium in the simplex [5,6]. Since $\mathcal{P} \cap \mathcal{O} = \emptyset$, neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ can have equilibria, so we must only construct affine feedbacks satisfying the invariance conditions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For the vertices of $\mathcal{S}_2$ we select control values $u_1 = (-5, 10)$, $u_3 = (-12, 10)$, $u_4 = (-12, 12)$, and $u_5 = (5, 12)$. Then we construct the unique affine feedback $u(x)$ on $\mathcal{S}_2$ satisfying $u(v_i) = u_i$, $v_i \in \mathcal{S}_2$ [1]. Similarly, we construct the affine feedback on $\mathcal{S}_1$ that achieves $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$. We conclude by Theorem 9 of [11] that the following discontinuous piecewise affine feedback solves