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Generalized Lyapunov function theorems and its applications in switched systems[☆]

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ABSTRACT

Generalized Lyapunov theorems of nonlinear systems are developed wherein all regularity assumptions on traditional Lyapunov function are removed. In particular, stability theorems of nonlinear systems are presented by replacing "V along the system trajectories is non-increasing" with "V along the system trajectories may increase its value during some proper time intervals". Furthermore, stability theorems of discrete-time and continuous-time switched system with unstable subsystems are derived, using the generalized Lyapunov theorems. Two numerical examples are included to illustrate the effectiveness of the method.

1. Introduction

Lyapunov function plays a vital role in stability theory and control theory, which is a scalar function that may be used to prove the stability of an equilibrium for ordinary differential equations (ODEs). The most important contribution to the stability theory of nonlinear systems is Lyapunov [1]. Informally, a Lyapunov function is a function that takes positive values everywhere except at the equilibrium in question, and decreases (or is non-increasing) along every trajectory of the ODE. It is well know that Lyapunov theory can make conclusions about trajectories of a system $\dot{x}(t) = f(x)$ without explicitly solving the differential equation and thus the principal merit of Lyapunov function-based stability analysis of ODEs is that the actual solution (whether analytical or numerical) of the ODE is not required [2,3]. In particular, Lyapunov's second method can provide the global and local stability results of an equilibrium for a nonlinear autonomous system if a smooth positive-definite function V can be constructed and the time rate of V at a neighborhood of the equilibrium is non-positive, with strict negative-definite ensuring asymptotic stability [4].

Most Lyapunov stability theorems require the Lyapunov function candidate V is a C^1 function and the derivative of V is negative-definite. However, due to the fact that system discontinuities cannot be avoided in the real world such as switched systems [5,6], discrete-event systems [7], and complex ecological systems [8] and so on, it is usually simpler to present discontinuous "Lyapunov function" to obtain the system stability [4]. On the other hand, the classical Lyapunov function requires V along the system trajectories is non-increasing, which restricts the application of the Lyapunov function theory. For example, Lyapunov function theory cannot be directly applied to switched system with unstable subsystems [6,9,10].

Switched systems, as one of the most important hybrid systems, are often used for modeling various control problems and some complex processes in engineering practice. In spite of their apparent simplicity, switched systems present a very complicated dynamical behaviors due to the multiple subsystems and various possible switching signals. For a large decade, the investigations of stability and stabilization for switched systems have attracted a growing attention in Systems Engineering and Computer Sciences communities, and fruitful results have been reported [5,6,9–22]. It is well known that ensuring independently stability for each mode of switched systems does not necessarily lead to global stability. As a matter of fact, a switched system with all stable subsystems may be unstable under unappropriate switching [5]. To investigate this problem, the multiple Lyapunov functions (MLFs) [5,10–17] and polytopic quadratic Lyapunov functions (PQLFs) [18,19] method were proposed, both of which are non-traditional Lyapunov functions. The reason for considering MLFs and PQLFs is that common Lyapunov function (CLF) does not exist for some switched systems. However, two methods usually require

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each modes stable. This property has been reported deeply by scientists. Until now, most results are based on all modes being individually stable. On the other hand, since unstable modes may appear inevitably and unavoidably in engineering practice while existing results cannot be straightly applied, switched system with unstable subsystems is not only academically challenging, but also of practical importance [6,10,20,21]. In the present of unstable subsystems in switched systems, the MLFs cannot be applied straightly, and have to be modified. Very recently, the literatures [20,21] obtained some results based on extended MLFs technique to deal with stabilization of switched systems with unstable subsystems.

In the present paper we develop generalized Lyapunov theorems of nonlinear systems wherein all regularity assumptions on traditional Lyapunov function are removed. In particular, stability theorems of nonlinear systems are presented by replacing "V along the system trajectories is non-increasing" with "V along the system trajectories may increase its value during some proper time intervals". Furthermore, based on the generalized Lyapunov theorems, stability theorems of discrete-time and continuous-time switched systems with unstable subsystems are derived. Compared with the extended MLFs technique in the literatures [20,21], our results do not require to construct a collection of Lyapunov-like functions, which is usually not an easy thing to do. Finally, two numerical examples are included to illustrate the effectiveness of the method.

2. Generalized Lyapunov theorem

Consider the following nonlinear system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0,\ t \geq 0 \tag{1}$$

and discrete-time system

$$x(t+1) = f(x(t)), \quad x(0) = x_0,\ t \in Z_0^+ \tag{2}$$

where $x(t) \in \mathfrak{D} \subseteq R^n$ denotes the system state vector, $\mathfrak{D}$ is an open set with $0 \in \mathfrak{D}, f : \mathfrak{D} \longrightarrow R^n$ and $f(0) = 0$. We assume $f(\cdot)$ is such that the system solution meets the existence and unique condition. R, Z^+ and Z_0^+ denote the set of real numbers, positive integers and non-negative integers, respectively.

The zero solution $x(t) \equiv 0$ to (1) (resp., (2)) is Lyapunov stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon,\ t \geq 0$ (resp., $t \in Z_0^+$). The zero solution to (1) (resp., (2)) is asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\longrightarrow+\infty} x(t) = 0$ [4]. The system (1) (resp., (2)) is called to be globally exponentially stable with stability degree $\lambda > 0$ if $\|x(t)\| \leq ce^{-\lambda t}\|x_0\|$ holds for all $t \in R_0^+$ (resp., $t \in Z_0^+$) and $x_0 \in R^n$ where $c \geq 1$ is a known constant.

Suppose a sequence $\alpha_n \geq 1,\ n \in Z_0^+$, satisfying $inf\{\alpha_n\} = \alpha \geq 1$, and a sequence $T_n \geq 0,\ n \in Z_0^+$, meeting $T_0 = 0, T_n < T_{n+1}, 2T_{n+1} \geq T_n + T_{n+2}$ with $T_n \longrightarrow +\infty$, as $n \longrightarrow +\infty$. Then there exists $m \in Z_0^+$ such that $t \in [T_m, T_{m+1})$ for every $t \in [0, +\infty)$.

Theorem 1 (*Generalized Lyapunov Theorem*)**.** *Assume that $V : \overline{U} \longrightarrow R$ is a continuous and positive-definite function and $\overline{U}$ is closed to the solution of the system, i.e., if $x_0 \in \overline{U}$, then $x(t) \in \overline{U},\ t \geq 0$. For any $x_0 \in \overline{U}$, if V satisfies*

(i) *for any $t \in [0, T_1)$, $V(x(t)) \leq MV(x_0)$;*
(ii) *for every $t \in [T_m, T_{m+1})$, $V(x(t)) \geq \alpha_n V(x(t + T_{m+1} - T_m))$,*

then the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is Lyapunov stable, where $\overline{U} \subset \mathfrak{D}$ and U is a bounded open set with $0 \in U$, $\overline{U}$ denotes the closure of U, $x(t)$ is the solution of the system (1) (resp., (2)), $M \geq 1$ is a certain scalar. The positive-definite function V satisfying the above conditions is called a generalized Lyapunov function (GLF).

Furthermore, if $\alpha > 1$ holds, the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is asymptotically stable.

Proof. Let $\varepsilon > 0$ be such that $B_\varepsilon = \{x|\|x\| \le \varepsilon\} \subset U$. Since $\overline{U} - B_\varepsilon$ is compact and V is continuous for $x \in \overline{U}$, it is true there exists $c = \min_{x\in\overline{U}-B_\varepsilon} V(x)$. It is easy to know $c > 0$ and $\{x|V(x) < l\} \subset \{x|V(x) < Ml\} \subset B_\varepsilon$ for all $l > 0$ satisfying $Ml < c$. Since V is a continuous and positive-definite function, it follows that there exists $\delta > 0$ such that $V(x) < l$, $\|x\| < \delta$. Thus, if $\|x_0\| < \delta$, then $V(x_0) < l$. Suppose $\|x_0\| < \delta$. For every $t \in [0, +\infty)$, without loss of generality, let $t \in [T_m, T_{m+1})$.

If $m = 0$, i.e., $t \in [0, T_1)$, it follows from (i) that $V(x(t)) < MV(x_0) < Ml$ and thus $x(t) \in B_\varepsilon$.

If $m > 0$, i.e., $t \in [T_m, T_{m+1})$, it follows from $2T_{n+1} \geq T_n + T_{n+2}$, $n \in Z_0^+$ that

$$t - T_m + T_{m-1} \in [T_{m-1}, T_m),$$
$$t - T_m + T_{m-2} \in [T_{m-2}, T_{m-1}), \dots, t - T_m + T_0 \in [0, T_1).$$

By (ii),

$$\begin{aligned} V(x(t)) &\leq \alpha_{m-1}^{-1} V(x(t - T_m + T_{m-1})) \\ &\leq (\alpha_{m-1}\alpha_{m-2})^{-1} V(x(t - T_m + T_{m-1} - T_{m-1} + T_{m-2})) \\ &= (\alpha_{m-1}\alpha_{m-2})^{-1} V(x(t - T_m + T_{m-2})) \\ &\leq \cdots \\ &\leq (\alpha_{m-1}\alpha_{m-2}\cdots\alpha_0)^{-1} V(x(t - T_m + T_0)) \\ &\leq \alpha^{-m} V(x(t - T_m)) \\ &\leq \frac{M}{\alpha^m} V(x_0) \\ &< \frac{Ml}{\alpha^m}. \end{aligned}$$

This is,

$$V(x(t)) < \frac{Ml}{\alpha^m}. \tag{3}$$

It is also true $x(t) \in B_\varepsilon$. Therefore, the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is Lyapunov stable.

Furthermore, if $\alpha > 1$ holds, it is easy to know from (3) that

$$V(x(t)) < \frac{Ml}{\alpha^m} \longrightarrow 0, \quad m \longrightarrow +\infty,\ t \longrightarrow +\infty.$$

It follows from the continuity of V, $\lim_{t\longrightarrow+\infty} V(x(t)) = V(\lim_{t\longrightarrow+\infty} x(t)) = 0$.

That is, $\lim_{t\longrightarrow+\infty} x(t) = 0$. Hence, the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is asymptotically stable for $\alpha > 1$. □

In particular, if $\alpha_n \equiv \alpha \geq 1, T_{n+1} - T_n \equiv T \geq 0, n \in Z_0^+$, Theorem 1 still holds.

Corollary 1. *Assume that $V : \overline{U} \longrightarrow R$ is a continuous and positive-definite function and $\overline{U}$ is closed to the solution of the system. If V satisfies*

(i) *for every $t \in [0, T)$, $V(x(t)) \leq MV(x_0)$;*
(ii) *for every $t \geq T$, $V(x(t)) \geq \alpha V(x(t + T))$,*

then the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is Lyapunov stable, where $\overline{U}$, $x(t)$ and M are defined in Theorem 1.

Furthermore, if $\alpha > 1$ holds, the zero solution $x(t) \equiv 0$ of the system (1) (resp., (2)) is asymptotically stable.

Remark 1. The relationship between Lyapunov function and GLF. Lyapunov function needs that the derivative of V exists with $\dot{V} \leq 0$ and thus $V(x(t)) \geq V(x(t+T))$ for any $T > 0$. From this point, the Lyapunov function is a special case of GLF. GLF does not require the existence of derivative of V, so its condition is much weaker. In addition, the GLF does not require V along the system trajectories monotonically decreasing, and it allows there exists proper increasing case for V along the system trajectories.