Applied Acoustics 108 (2016) 59–62

Contents lists available at ScienceDirect

Applied Acoustics

journal homepage: www.elsevier.com/locate/apacoust

Discontinuous Galerkin method coupled with an integral representation for solving the three-dimensional time-harmonic Maxwell equations

Nabil Gmati [a], Stéphane Lanteri [b], Anis Mohamed [a,*]

[a] National Engineering School of Tunis, ENIT-LAMSIN BP 37, 1002 Tunis, LR 99-ES-20, Tunisia
[b] INRIA, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex, France

ARTICLE INFO

Article history:
Received 2 May 2015
Received in revised form 4 August 2015
Accepted 16 August 2015
Available online 7 September 2015

Keywords:
Maxwell equations
Time-harmonic
Discontinuous Galerkin method
Integral representation

ABSTRACT

In this paper, we present a mathematical and numerical studies of the three-dimensional time-harmonic Maxwell equations. The problem is solved by a discontinuous Galerkin DG method coupled with an integral representation. This study was completed by some numerical tests to justify the effectiveness of the proposed approach. The numerical simulation was done by an iterative solver implemented in FORTRAN.

1. Introduction

Electromagnetic phenomena are generally described by the electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ which are related by the following Maxwell equations:

$$\begin{cases} -\varepsilon\partial_t\mathcal{E} + curl\mathcal{H} = 0 \\ \mu\partial_t\mathcal{H} + curl\mathcal{E} = 0 \end{cases} \tag{1}$$

where ε and μ are the complex-valued relative dielectric permittivity and the relative magnetic permeability, respectively. In the presence of an obstacle D, we are interested in particular Solutions of the Maxwell's equations assuming a time-harmonic regime:

$$\begin{cases} \mathcal{E}(x,t) = Re(E(x)exp(-i\omega t)) \\ \mathcal{H}(x,t) = Re(H(x)exp(-i\omega t)) \end{cases}$$

where E, H are two complex values and ω denotes the angular frequency. The time-harmonic Maxwell system is then written as follows:

$$\begin{cases} curlE - i\omega\mu H = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\ curlH + i\omega\varepsilon E = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \end{cases} \tag{2}$$

The proposed idea to solve this problem is to limit the domain, which is initially unbounded, by a fictitious boundary Γ_a on which we impose an absorbing boundary condition defined in terms of an integral representation (RI) of the solution.

This concept was introduced by Lenoir and Jami in hydrodynamics in 1978 [1], then in 1996 by Lenoir and Hazard for the Maxwell's equations by using nodal finite elements [2]. Liu and Jin presented a very interesting results in 3D by proposing an iterative algorithm which was then interpreted as a Schwartz technique with total recovery by Ben Belgacem et al. in [3]. El Bouajaji and Lanteri have used in [4] a discontinuous Galerkin methods to solve the two-dimensional time-harmonic Maxwell's equations. This work was extended to solve the three-dimensional time-harmonic Maxwell's equations in [5,6].

Our objective in this paper is to study and implement the coupling between a DG method an integral representation of the solution imposed through a Silver-Müller absorbing boundary condition. It is an extension of the work in the two-dimensional case by El Bouajaji et al. [7] (see Fig. 1).

Ω is the limited domain by Γ_m the boundary of the obstacle D and the fictitious boundary denoted Γ_a.

On Γ_m, we take the perfect conductor condition.

* Corresponding author.
E-mail address: midani.anis@gmail.com (A. Mohamed).

http://dx.doi.org/10.1016/j.apacoust.2015.08.004
0003-682X/

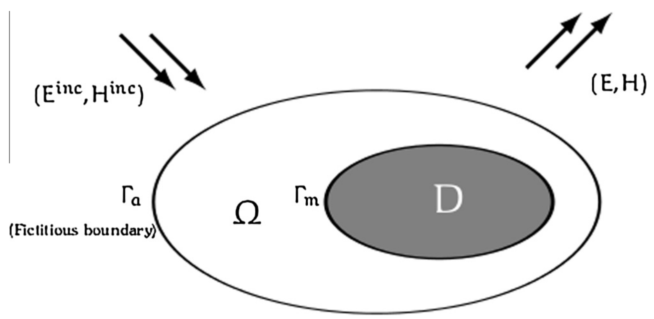

Fig. 1. Diffraction of an electromagnetic wave in the presence of an obstacle D.

2. Formulation of the problem

Find $E, H \in H(curl, \Omega_i)$ such as:

$$\begin{cases} i\omega\varepsilon E - curl H = 0 & \text{in} \quad \mathbb{R}^3 \\ i\omega\mu H + curl E = 0 & \text{in} \quad \mathbb{R}^3 \\ n \wedge E = -n \wedge E^{inc} & \text{on} \quad \Gamma_m \\ n \wedge E - Z \cdot n \wedge (n \wedge H) = n \wedge \Re(E) - Z \cdot n \wedge (n \wedge \Re(H)) & \text{on} \quad \Gamma_a \end{cases} \tag{3}$$

where

- ω is the angular frequency of the problem,
- n is a normal vector to the boundary Γ_a,
- $Z = \sqrt{\mu/\varepsilon}$,
- $\Re(E)$ and $\Re(E)$ are the expression of the electric and magnetic fields E and H, respectively, on Γ_a. They are given by the following integral representation [13]:

$$\Re(E) = curl_x \int_{\Gamma_m} n(y) \wedge E(y) G(x,y) \partial\sigma_y - (1/i\omega\mu) curl_x curl_x \int_{\Gamma_m} n(y) \wedge H(y) G(x,y) \partial\sigma_y$$

$$\Re(H) = curl_x \int_{\Gamma_m} n(y) \wedge H(y) G(x,y) \partial\sigma_y + (1/i\omega\varepsilon) curl_x curl_x \int_{\Gamma_m} n(y) \wedge E(y) G(x,y) \partial\sigma_y$$

- $G(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$, $x \neq y$, is the fundamental solution of the Helmholtz equation.

We denote by (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$. For simplicity, we denote by W the vector $\begin{bmatrix} E \\ H \end{bmatrix}$.

Let G_l, for $l \in \{1,2,3\}$, be the matrix defined as follows:

$$G_l = \begin{bmatrix} 0_{3\times3} & N_{e_l} \\ N_{e_l}^t & 0_{3\times3} \end{bmatrix}$$

where for a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $N_v = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}$.

Therefore, system (3) can be rewritten in the following conservative form:

$$\begin{cases} i\omega Q W + G_1 \partial_x W + G_2 \partial_y W + G_3 \partial_z W = 0 & \text{on} \quad \Omega \\ (M_{\Gamma_m} - G_n) \cdot (W + W^{inc}) = 0 & \text{in} \quad \Gamma_m \\ (M_{\Gamma_a} - G_n) \cdot (W - \Re(W)) = 0 & \text{in} \quad \Gamma_a \end{cases} \tag{4}$$

where

- $G_n = G_1 n_1 + G_2 n_2 + G_3 n_3$,
- $\Re(W) = \begin{bmatrix} \Re(E) \\ \Re(H) \end{bmatrix}$, $Q = \begin{bmatrix} \varepsilon \cdot I_3 & 0_{3\times3} \\ 0_{3\times3} & \mu \cdot I_3 \end{bmatrix}$ and $M_{\Gamma_m} = \begin{bmatrix} 0_{3\times3} & N_n \\ -N_n^t & 0_{3\times3} \end{bmatrix}$,
- $M_{\Gamma_a} = |G_n| = G_n^+ - G_n^-$.[1]

3. Discretization

We decompose the domain Ω_h into N tetrahedral cells K, we denote by τ_h the set of these elements. We are looking an approximate solution $W_h = \begin{bmatrix} E_h \\ H_h \end{bmatrix}$ in $V_h \times V_h$ where:

$V_h = \{W \in [L^2(\Omega)]^3 / W_{\backslash K} \in P_p(K)\}$ defined a functional space

where, over an element K, $P_p(K)$ denotes the space of vectors with components polynomial of degree at most p.

Also we note:

$$\Gamma^0 = \bigcup_{K,\widetilde{K} \in \tau_h} \overline{K} \cap \overline{\widetilde{K}}, \quad \Gamma^m = \bigcup_{K \in \tau_h} \overline{K} \cap \Gamma_m \quad \text{and} \quad \Gamma^a = \bigcup_{K \in \tau_h} \overline{K} \cap \Gamma_a$$

Multiplying the first equation in (4) by $V \in V_h \times V_h$ and integrating over Ω_h, we get that W_h solves

$$\int_{\Omega_h} (i\omega Q W_h)^t \cdot \overline{V} dx + \int_{\Omega_h} (\nabla \cdot F(W)_h)^t \cdot \overline{V} dx = 0 \tag{5}$$

where $F(W) = (F_1(W), F_2(W), F_3(W))$ is a linear mapping from $\mathbb{R}^6$ to $\mathbb{R}^6 \times \mathbb{R}^6 \times \mathbb{R}^6$, such: $F_1(W) = G_1 W$; $F_2(W) = G_2 W$; $F_3(W) = G_3 W$.

Using Green formula we obtain:

$$\int_{\Omega_h} (i\omega Q W_h)^t \cdot \overline{V} dx - \sum_{K \in \tau_h} \left(\int_K (F(W)_h)^t \cdot \nabla \overline{V} dx + \int_{\partial K} (F(W)_h \cdot n) \cdot \overline{V} \partial\sigma \right) = 0 \tag{6}$$

Using the same techniques adopted by Ern and Guermond [8,9], by summing over elements, we obtain the following formulation: $\forall V \in V_h \times V_h$, K an element of τ_h:

$$\begin{aligned} &\text{Find} \quad W_h \in V_h \times V_h \quad \text{such as :} \\ &\int_{\Omega_h} (i\omega Q W_h)^t \overline{V} dx - \sum_{K \in \tau_h} \int_K ((F(W))_h)^t \cdot \nabla \overline{V} dx \\ &+ \sum_{F \in \Gamma^0} \int_F \left[(S_F \cdot [\![W_h]\!])^t [\![\cdot \overline{V}]\!] - (G_{n_F} \cdot [\![W_h]\!])^t \cdot \{\overline{V}\} \right] \partial\sigma \\ &+ \sum_{F \in \Gamma^a} \left[\int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{n_F}) W_h \right)^t \overline{V} \partial\sigma \right. \\ &\left. + \int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{n_F}) \Re(W) \right)^t \overline{V} \partial\sigma \right] \\ &+ \sum_{F \in \Gamma^m} \int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{n_F}) W_h \right)^t \overline{V} \partial\sigma \\ &= \sum_{F \in \Gamma^m} \int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{n_F}) W_h^{inc} \right)^t \overline{V} \partial\sigma \end{aligned}$$

where $\Re(W) = \int_\Gamma K(x,y) W(y) \partial\sigma_y$. We also define the jump and average of a vector V to $V_h \times V_h$ on one face F shared between two elements K and $\widetilde{K}$, respectively, as follows:

[1] If $P\Lambda P^{-1}$ is the natural factorization of G_n then $G_n^{\pm} = P\Lambda^{\pm}P^{-1}$ where Λ^+ (resp. Λ^-) includes only positive eigenvalues (resp. negative).