Operations Research Letters 46 (2018) 389–392

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On the complete set packing and set partitioning polytopes: Properties and rank 1 facets

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ARTICLE INFO

Article history:
Received 23 June 2017
Received in revised form 11 April 2018
Accepted 24 April 2018
Available online 5 May 2018

Keywords:
Set packing
Set partitioning
Polyhedral combinatorics
Rank 1 cuts
Facets

ABSTRACT

This paper studies two polytopes: the complete set packing and set partitioning polytopes, which are both associated with a binary n-row matrix having all possible columns. Cuts of rank 1 for the latter polytope play a central role in recent exact algorithms for many combinatorial problems, such as vehicle routing. We show the precise relation between the two polytopes studied, characterize the multipliers that induce rank 1 clique facets and give several families of multipliers that yield other facets.

1. Introduction

Let A be a binary matrix with n rows and m columns. The set packing polytope associated with A, denoted as $SPP_{\leq}(A)$, is defined as the convex hull of the integer solutions to the system $Ax \leq \mathbb{1}$, where $\mathbb{1}$ represents the n-dimensional all-ones vector and $x \in \{0, 1\}^m$. Such solutions are equivalent to stable sets of the intersection graph G derived from A, i.e., a graph whose vertices represent columns of A and whose edges indicate non-orthogonality between the corresponding columns. Therefore, we also denote this polytope as $SPP_{\leq}(G)$. A closely related polytope is the set partitioning polytope, defined analogously with respect to the system $Ax = \mathbb{1}$.

Since the seventies, many authors have studied $SPP_{\leq}(G)$, proposing facet-inducing inequalities associated with specific graphs: cliques [19], odd holes [19], odd anti-holes [18], webs [26], anti-webs [26], $K_{1,3}$-free graphs [13], wheels [9], antiweb-wheels [10] and grilles [7]. For some graphs, such as perfect graphs, series–parallel graphs and graphs that do not have a 4-wheel as a minor, complete characterizations of $SPP_{\leq}(G)$ are known [5,6,12,16]. Facet-inducing inequalities with binary coefficients were studied in [3,12]. Facet-generating procedures for $SPP_{\leq}(G)$ were described in [4,5,7,12,18,20,24,27]. In contrast, the set partitioning polytope is seldom studied in the literature (see for instance [1,2,25]) due to its more complex structure—even computing its dimension is a NP-Hard problem.

In this paper, we study the complete set packing polytope $CSPP_{\leq}(n)$ and the complete set partitioning polytope $CSPP_{=}(n)$, which are both associated with a binary n-row matrix A having all possible $(2^n - 1)$ non-zero columns. While $CSPP_{=}(n)$ has already been defined in [23], as far as we know, $CSPP_{\leq}(n)$ is studied for the first time in this paper. However, some algorithms for the complete set packing problem have been proposed in the literature (see, for instance, [17,28]). The definitions for both polytopes are formalized as follows.

$$CSPP_{=}(n) = \text{Conv}\left\{\sum_{j=1}^{2^n-1} b^j\lambda_j = \mathbb{1}, \lambda \in \{0,1\}^{2^n-1}\right\}$$

$$CSPP_{\leq}(n) = \text{Conv}\left\{\sum_{j=1}^{2^n-1} b^j\lambda_j \leq \mathbb{1}, \lambda \in \{0,1\}^{2^n-1}\right\}$$

where b^j is the column associated with the binary representation of j. For example, if $n = 3$, then $b^1 = (1, 0, 0)^T$, $b^2 = (0, 1, 0)^T$, $b^3 = (1, 1, 0)^T$, etc. A column $e^i = b^{2^{i-1}}$, $1 \leq i \leq n$, is the *singleton* column associated to row i. Let $\bar{b}^j = \mathbb{1} - b^j = b^{2^n-1-j}$ be the complement of b^j. Also, we define $B = \bigcup_{j=1}^{2^n-1} b^j$, the set of all columns, and we denote a solution to the complete set partitioning (set packing) problem by a subset s of columns whose incidence vector belongs to $CSPP_{=}(n)$ ($CSPP_{\leq}(n)$). The incidence vector of a solution s is the

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https://doi.org/10.1016/j.orl.2018.04.006
0167-6377/

vector $\sum_{b^j \in S} \hat{e}^j$, where $\hat{e}^j$ is the $(2^n - 1)$–dimensional unitary vector such that $\hat{e}^j_j = 1$.

Our study of the complete polytopes is motivated by the many applications that can be modeled as set packing/partitioning problems with a very large number of columns, where explicit representation of the coefficient matrix is practically impossible and the linear relaxations have to be solved by column generation. In this context, any cutting plane should have a well-defined coefficient for every possible column, since it is not possible to predict which columns will be generated. In other words, cuts should be valid for $CSPP_=(n)/CSPP_\leq(n)$.

A typical such application is the vehicle routing problem (VRP), where one looks for a minimum cost set of routes that serve a set of customers $\mathcal{C}$. These routes should respect operational constraints, that vary according to the considered variant. Let Ω, c_r and a^r_i denote, respectively, the set of feasible routes, the cost of route r, and the number of times route r visits customer i. The set partitioning formulation of the VRP follows:

$$\min \sum_{r \in \Omega} c_r \lambda_r \tag{1}$$

$$\text{s.t.} \quad \sum_{r \in \Omega} a^r_i \lambda_r = 1, \qquad \forall i \in \mathcal{C}, \tag{2}$$

$$\lambda_r \in \{0, 1\}, \qquad \forall r \in \Omega. \tag{3}$$

State-of-the-art exact algorithms for many VRPs, including its most classical variants, the Capacitated VRP (CVRP) and the VRP with Time Windows (VRPTW), are based on a combination of column and cut generation over the above formulation. However, the addition of general cuts for $CSPP_=(n)$ has the serious drawback of complicating a lot the pricing problem (i.e., the column generation subproblem), making the algorithm unpractical. Jepsen et al. [15] realized that some cuts with Chvátal–Gomory rank 1 could be better treated in the pricing. Recently, Pecin et al. [22] introduced the so-called limited-memory technique, for further minimizing the impact in the pricing of rank 1 cuts. This led to big improvements in the performance of exact algorithms for CVRP [22] and VRPTW [21], more than doubling the size of the instances that can be solved. This motivated Pecin et al. [23] to determine computationally 9 sets of rational multipliers that are capable of generating all cuts of rank 1 that induce facets of $CSPP_=(n)$, for $n \leq 5$. The authors argued that the new multipliers contributed decisively for solving a previously open CVRP instance with 420 customers, the largest classical instance ever solved. However, that computational "brute force" approach breaks down for $n > 5$.

This paper is a theoretical analysis of $CSPP_=(n)$, aimed at finding infinite families of multipliers that produce facets for arbitrarily large values of n. First, we prove a very strong relationship between $CSPP_=(n)$ and $CSPP_\leq(n)$ (Section 2). More specifically, we show that, with very few exceptions, every facet-inducing inequality for the former polytope is also facet-inducing for the latter polytope, and vice-versa. This essentially means that the study of $CSPP_=(n)$ can be reduced to the study of the simpler $CSPP_\leq(n)$ polytope. Second, we characterize a set of multipliers that can induce all rank 1 maximal clique inequalities for those polytopes (Section 3.1). Finally, we propose 7 families of multipliers that generalize the 9 sets of multipliers found by Pecin et al. [23] and prove that they are facet-defining (Section 3.2).

2. Properties

Definition 1. The intersection graph associated with $CSPP_=(n)$ and $CSPP_\leq(n)$, denoted as G_n, is a graph where each vertex represents a column b^j and an edge exists iff the columns represented by its endpoints are non-orthogonal.

For a set of columns $S \subseteq B$, we define $\perp(S)/\not\perp(S)$ as the set of all singleton columns that are orthogonal /non-orthogonal to all columns in S. Let also $\mathcal{K}_=$ be the set of the n maximal cliques of G_n that contain singleton columns and let $\mathcal{K}_\leq$ be the set of all other maximal cliques of G_n. For any maximal clique $K \in \mathcal{K}_= \cup \mathcal{K}_\leq$, the maximal clique inequality $\sum_{v \in K} \lambda_v \leq 1$ is valid for both $CSPP_=(n)$ and $CSPP_\leq(n)$. Moreover, the cliques in $\mathcal{K}_=$ define the n equalities of $CSPP_=(n)$, that is, any point $\lambda \in CSPP_=(n)$ satisfies $\sum_{v \in K} \lambda_v = 1$, for all $K \in \mathcal{K}_=$.

Lemmas 1 to 3 state some basic properties of $CSPP_=(n)$ and $CSPP_\leq(n)$.

Lemma 1. *$CSPP_\leq(n)$ is full-dimensional and the dimension of $CSPP_=(n)$ is $2^n - n - 1$.*

Proof. Since $SPP_\leq(A)$ is full-dimensional [2], $CSPP_\leq(n)$ is also full-dimensional. For $CSPP_=(n)$, consider the incidence vectors of the following $2^n - n$ solutions:

- $s_j = \{b^j\} \cup \perp(\{b^j\})$, $1 \leq j \leq 2^n - 1$ and b^j is not a singleton column,
- $s = \bigcup_{k=1}^n \{e^k\}$,

which are clearly linearly independent. □

Lemma 2. *Let $a^T\lambda \leq a_0$ be facet-inducing for $CSPP_=(n)$. For every column b^j, it holds that $a_j \geq \sum_{b^i \in \not\perp(\{b^j\})} a_i$.*

Proof. Since $a^T\lambda \leq a_0$ induces a facet, there is a set of columns S such that $\{b^j\} \cup S$ is a feasible solution and $a_j + \sum_{b^i \in S} a_i = a_0$, otherwise a_j could be increased without cutting off any feasible solution, which is a contradiction to the facet-defining property of $a^T\lambda \leq a_0$. Hence, if $a_j < \sum_{b^i \in \not\perp(\{b^j\})} a_i$, the feasible solution $\not\perp(\{b^j\}) \cup S$ would be cut off by $a^T\lambda \leq a_0$. □

Lemma 3. *Any inequality $a^T\lambda \leq a_0$ that is facet-inducing for $CSPP_=(n)$ can be rewritten as $\alpha^T\lambda \leq \alpha_0$ such that: $\alpha_j = 0$ for every singleton column b^j; $\alpha_j \geq 0$ for every column b^j; and $\alpha_0 \geq 0$. Moreover, in this form, such a facet is a valid inequality for $CSPP_\leq(n)$.*

Proof. This proof is algorithmic. Start with $\alpha_j = a_j$ for $j = 1, \ldots, 2^n - 1$. For each singleton column b^j, do the following. If $\alpha_j > 0$, subtract α_j times the equality containing b^j from $\alpha^T\lambda \leq \alpha_0$. If $\alpha_j < 0$, add $|\alpha_j|$ times the equality containing b^j to $\alpha^T\lambda \leq \alpha_0$. Now that $\alpha_j = 0$ for every singleton column b^j, it follows from Lemma 2 that $\alpha_i \geq 0$ for every column b^i. Moreover, $\alpha_0 \geq 0$, otherwise any feasible solution would be cut off.

Now, it remains to prove that $\alpha^T\lambda \leq \alpha_0$ is satisfied by any vertex λ of $CSPP_\leq(n)$. For that, let λ' be the vertex of $CSPP_=(n)$ obtained from λ by increasing the coordinates of singleton columns until $\sum_{j \in K} \lambda'_j = 1$ for all $K \in \mathcal{K}_=$. Since $\alpha_j = 0$ for every singleton column b^j, we have that $\alpha^T\lambda = \alpha^T\lambda' \leq \alpha_0$. □

Next, we introduce some facet-inducing inequalities for $CSPP_=(n)$ that are necessary to establish the main property.

Lemma 4. *The non-negativity inequality $\lambda_j \geq 0$ defines a facet of $CSPP_=(n)$ iff b^j is not a singleton column.*

Proof. If b^j is not a singleton column, then the incidence vectors of the following $2^n - n - 1$ solutions satisfy $\lambda_j \geq 0$ at equality and are clearly linearly independent:

- $s_i = \{b^i\} \cup \perp(\{b^i\})$, $1 \leq i \leq 2^n - 1$, $b^i \neq b^j$ and b^i is not a singleton column,
- $s = \bigcup_{k=1}^n \{e^k\}$.