Operations Research Letters 46 (2018) 414–417

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

# A note on capacity models for network design

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ARTICLE INFO

*Article history:*
Received 28 November 2017
Received in revised form 16 March 2018
Accepted 8 May 2018
Available online 22 May 2018

*Keywords:*
Valid inequalities
Network design
Capacity
Traffic

ABSTRACT

In network design problems, capacity constraints are modeled in three different ways depending on the application: *directed*, *bidirected* and *undirected*. In the literature, polyhedral investigations for strengthening mixed-integer formulations are done separately for each model. In this note, we examine the relationship between these models to provide a unifying approach and show that one can indeed translate valid inequalities from one to the others.



## 1. Introduction

In network design problems, capacity constraints are modeled in three different ways depending on the application and the underlying technology for installing capacity: *directed*, *bidirected*, and *undirected*. In *directed* models, the total flow on an arc is limited by the capacity of the directed arc. In *bidirected* models, if a certain capacity is installed on an arc, then the same capacity also needs to be installed on the reverse arc. Whereas in *undirected* models, the sum of the flow on an arc and its reverse arc is limited by the capacity of the undirected edge associated with the two arcs.

In the literature, polyhedral investigations for strengthening mixed-integer formulations are done separately for each model. Undirected capacity models are considered in [2,10,11,13–16]. Bidirected capacity models are studied in [7,9]. Whereas directed models are considered in [1,3–6,8,12]. Oriolo [16] gives a characterization of domination between symmetric traffic matrices for the undirected capacity model. In this paper, we examine the relationship between these three separately-studied models to provide a unifying approach and show how one can translate valid inequalities from one to the others. In particular, we show that the projections of the undirected and bidirected models onto the capacity variables are the same. We demonstrate that valid inequalities previously given for undirected and bidirected models can be derived as a consequence of the relationship between these models and the directed model.

Let $G = (N, E)$ be an undirected graph with node set $N$ and edge set $E$. Let $A$ be the arc set obtained from $E$ by including the arcs in each direction for the edges in $E$. Let $M$ denote the set of facility types where a unit of facility $m \in M$ provides capacity $c_m$. Depending on the model, facilities are installed either on the edges or on the arcs of the network and accordingly, we use $\bar{c} \in \mathbb{R}^E$ or $\bar{c} \in \mathbb{R}^A$ to denote the existing capacities on the edges or arcs of the network. Without loss of generality, we assume that $c_m \in \mathbb{Z}$ for all $m \in M$ and $c_1 < c_2 < \cdots < c_{|M|}$. Let the demand data for the problem be given by the square matrix $T = \{t_{ij}\}$, where $t_{ij} \geq 0$ is the amount of directed traffic that needs to be routed from node $i \in N$ to $j \in N$. Let $K = \{(ij) \in N \times N \;:\; i \neq j\}$ and define the $|K| \times |N|$ demand matrix $D = \{d_u^k\}$, where

$$d_u^k = \begin{cases} t_{ij} & \text{if } u = j, \\ -t_{ij} & \text{if } u = i, \\ 0 & \text{otherwise,} \end{cases}$$

for $(ij) = k \in K$ and $u \in N$.

## 2. Undirected capacity model

In the undirected network design problem, the sum of the flow on an arc and its reverse arc is limited by the capacity of the undirected edge associated with the two arcs. This problem can be formulated as follows:

$$\min f(x, y)$$

$$\text{s.t.} \sum_{j \in N_i^+} x_{ij}^k - \sum_{j \in N_i^-} x_{ji}^k = d_i^k, \quad \text{for } k \in K,\ i \in N, \tag{1}$$

$$\sum_{k \in K} x_{ij}^k + \sum_{k \in K} x_{ji}^k \leq \bar{c}_e + \sum_{m \in M} c_m y_{m,e}, \quad \text{for } e = \{i, j\} \in E, \tag{2}$$

$$x,\ y \geq 0, \quad y \in \mathbb{Z}^{M \times E}, \tag{3}$$

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https://doi.org/10.1016/j.orl.2018.05.002
0167-6377/

where $x$ and $y$ denote the flow and capacity variables, respectively, and $f$ denotes the cost function. In most applications, the function $f$ can be decomposed as $f(x, y) = f^1(x) + f^2(y)$ and, furthermore, it is typically linear.

Let $U(T)$ denote the set of feasible solutions to inequalities (1)–(3). A capacity vector $\bar{y}$ *accommodates* traffic $T$ if there exists a feasible flow vector $x$ such that $(x, \bar{y}) \in U(T)$. In other words, $\bar{y}$ accommodates $T$ if $\bar{y} \in \text{proj}_y(U(T))$, where $\text{proj}_y(\cdot)$ denotes the orthogonal projection operator onto the space of the $y$ variables. We say that two traffic matrices $T$ and $\widehat{T}$ are *pairwise similar* if $t_{ij} + t_{ji} = \hat{t}_{ij} + \hat{t}_{ji}$ for all $(ij) \in K$. We next show that $\text{proj}_y(U(T)) = \text{proj}_y(U(\widehat{T}))$ provided that $T$ and $\widehat{T}$ are pairwise similar.

**Lemma 1.** *Capacity vector $y$ accommodates $T$ if and only if it accommodates all $\widehat{T}$ pairwise similar to $T$.*

**Proof.** Let $T$ and $\widehat{T}$ be pairwise similar. Consider a pair of nodes $u, v \in N$ with nonzero traffic, i.e., $\sigma = t_{uv} + t_{vu} = \hat{t}_{uv} + \hat{t}_{vu} > 0$.

Assuming $y$ accommodates $T$, let $x$ be a flow vector such that $(x, y) \in U(T)$. Construct $\hat{x}$ from $x$ by letting all entries of $\hat{x}$ corresponding to $k \in K \setminus \{uv, vu\}$ be same as that of $x$. Let $\alpha = \hat{t}_{uv}/\sigma$. For the remaining entries of $\hat{x}$ associated with commodities $uv$ and $vu$, we set $\hat{x}^{uv}_{ij} = \alpha(x^{uv}_{ij} + x^{vu}_{ji})$ and $\hat{x}^{vu}_{ij} = (1-\alpha)(x^{vu}_{ij} + x^{uv}_{ji})$ for all $(ij) \in A$.

Notice that $\hat{x}^{vu}_{ij} + \hat{x}^{vu}_{ji} + \hat{x}^{uv}_{ij} + \hat{x}^{uv}_{ji} = x^{vu}_{ij} + x^{vu}_{ji} + x^{uv}_{ij} + x^{uv}_{ji}$ for all $\{i, j\} \in E$. Therefore, $\hat{x}$ satisfies the capacity constraints (2). In addition, $\hat{x}$ also satisfies the flow balance constraints (1) as $\hat{d}^{uv}_i = \alpha(d^{uv}_i + d^{vu}_i)$ and $\hat{d}^{vu}_i = (1-\alpha)(d^{uv}_i + d^{vu}_i)$ for all $i \in N$. Repeating the same argument for the remaining pairs of nodes proves the claim. □

Lemma 1 can also be shown using the metric inequalities as done in [16].

**Definition 1.** An objective function $f$ is called *arc-symmetric* if $f(x, y) = f(\hat{x}, y)$ whenever

$$x^{vu}_{ij} + x^{vu}_{ji} + x^{uv}_{ij} + x^{uv}_{ji} = \hat{x}^{vu}_{ij} + \hat{x}^{vu}_{ji} + \hat{x}^{uv}_{ij} + \hat{x}^{uv}_{ji}$$

for $uv, vu \in K$ and $\{i, j\} \in E$.

**Lemma 2.** *Let $T$ and $\widehat{T}$ be pairwise similar and $f$ be arc-symmetric. If $(x, y) \in U(T)$, then there exists $(\hat{x}, y) \in U(\widehat{T})$ such that $f(x, y) = f(\hat{x}, y)$.*

**Proof.** As in the proof of Lemma 1, it is possible to construct a flow vector $\hat{x}$ such that $(\hat{x}, y) \in U(\widehat{T})$. Furthermore, as $\hat{x}^{vu}_{ij} + \hat{x}^{vu}_{ji} + \hat{x}^{uv}_{ij} + \hat{x}^{uv}_{ji} = x^{vu}_{ij} + x^{vu}_{ji} + x^{uv}_{ij} + x^{uv}_{ji}$ for all $(ij) \in A$ and the function is arc-symmetric, the result follows. □

Given a traffic matrix $T$, we define its symmetric counterpart to be $T^* = (T + T^T)/2$. In other words, $t^*_{uv} = t^*_{vu} = (t_{uv} + t_{vu})/2$. Also note that $T$ and $T^*$ are pairwise similar. We have so far established that optimizing an arc-symmetric cost function $f$ over $U(T)$ is equivalent to optimizing it over $U(T^*)$.

**Lemma 3.** *Let $f$ be an arc-symmetric function and $T^*$ be a symmetric matrix. If $(x, y) \in U(T^*)$, then there exists $(\hat{x}, y) \in U(T^*)$ such that $\hat{x}^{uv}_{ij} = \hat{x}^{vu}_{ji}$ for all $(ij) \in A$ and $(uv) \in K$. Furthermore, $f(x, y) = f(\hat{x}, y)$.*

**Proof.** Let $\tilde{x}$ be such that $\tilde{x}^{uv}_{ij} = x^{vu}_{ji}$ for all $(ij) \in A$ and $(uv) \in K$. As $T^*$ is symmetric, $(\tilde{x}, y) \in U(T^*)$. Furthermore, by convexity, defining $\hat{x} = (x+\tilde{x})/2$ we have $(\hat{x}, y) \in U(T^*)$. In addition, $\hat{x}^{uv}_{ij} = \hat{x}^{vu}_{ji}$ for all $(ij) \in A$ and $(uv) \in K$. Finally, as $f$ is arc-symmetric and $\hat{x}^{vu}_{ij} + \hat{x}^{vu}_{ji} + \hat{x}^{uv}_{ij} + \hat{x}^{uv}_{ji} = x^{vu}_{ij} + x^{vu}_{ji} + x^{uv}_{ij} + x^{uv}_{ji}$, for all $(ij) \in A$ and $(uv) \in K$, the claim holds. □

Let $U^=(T)$ denote the set of feasible solutions $(x, y)$ to constraints (1)–(3) together with the following equations

$$x^{uv}_{ij} = x^{vu}_{ji} \quad \text{for all } (ij) \in A,\ (uv) \in K. \tag{4}$$

Lemma 3 in conjunction with Lemma 2 establishes that when $f$ is an arc-symmetric function, optimizing $f$ over $U(T)$ is same as optimizing it over $U^=(T^*)$.

**Proposition 1.** *Let $f$ be an arc-symmetric function, $T$ be a traffic matrix and let $T^*$ be its symmetric counterpart. Then*

$$\min_{(x,y)\in U(T)} f(x, y) = \min_{(x,y)\in U(T^*)} f(x, y) = \min_{(x,y)\in U^=(T^*)} f(x, y).$$

*Furthermore, given an optimal solution to any one of the problems, optimal solutions to the other two can be constructed easily.*

Furthermore, notice that if (4) holds, then

$$\sum_{k\in K} x^k_{ij} + \sum_{k\in K} x^k_{ji} = 2\max\Big\{\sum_{k\in K} x^k_{ij}, \sum_{k\in K} x^k_{ji}\Big\} \tag{5}$$

for all $(ij) \in A$. We next relate these observations on undirected capacity models to network design problems with bidirected capacity constraints.

## 3. Bidirected capacity model

In the bidirected network design problem, the total flow on an arc and total flow on its reverse arc are each limited by the capacity of the undirected edge associated with these arcs. This problem can be formulated as follows:

$$\begin{aligned}
& \min\ f(x, y) \\
\text{s.t.}\ & \sum_{j\in N^+_i} x^k_{ij} - \sum_{j\in N^-_i} x^k_{ji} = d^k_i, \quad \text{for } k \in K,\ i \in N, && (6)\\
& \max\Big\{\sum_{k\in K} x^k_{ij}, \sum_{k\in K} x^k_{ji}\Big\} \le \bar{c}_e + \sum_{m\in M} c_m y_{m,e}, \quad \text{for } e = \{i, j\} \in E, && (7)\\
& x,\ y \ge 0, \quad y \in \mathbb{Z}^{M\times E}. && (8)
\end{aligned}$$

Let $B(T)$ be the set of feasible solutions $(x, y)$ to inequalities (7)–(8). We next show that if $T$ is symmetric and $(x, y) \in B(T)$, then there exists a flow vector $\hat{x}$ such that $(\hat{x}, y) \in B(T)$ and $\hat{x}$ satisfies (4). Furthermore if $f$ is an arc-symmetric cost function, then $f(x, y) = f(\hat{x}, y)$.

**Lemma 4.** *Let $f$ be an arc-symmetric function and $T^*$ be a symmetric matrix. If $(x, y) \in B(T^*)$, then there exists $(\hat{x}, y) \in B(T^*)$ such that $\hat{x}^{uv}_{ij} = \hat{x}^{vu}_{ji}$ for all $(ij) \in A$ and $(uv) \in K$. Furthermore, $f(x, y) = f(\hat{x}, y)$.*

**Proof.** The proof is essentially identical to that of Lemma 3. First we construct $\tilde{x} \in U(T^*)$ by letting $\tilde{x}^{uv}_{ij} = x^{vu}_{ji}$ for all $(ij) \in A$ and $(uv) \in K$. Then we define $\hat{x} = (x + \tilde{x})/2$ and observe that $(\hat{x}, y) \in U(T^*)$ and that it satisfies the properties in the claim. □

Therefore, if $T^*$ is a symmetric traffic matrix, then

$$\min_{(x,y)\in B(T^*)} f(x, y) = \min_{(x,y)\in B^=(T^*)} f(x, y), \tag{9}$$

where $B^=(T^*)$ is the set of feasible solution $(x, y)$ to constraints (6)–(8) together with Eq. (4). We next show that optimizing an arc-symmetric cost function over $U(T)$ is equivalent to optimizing a slightly different function over $B(2T^*)$.

**Proposition 2.** *Let $f$ be an arc-symmetric function, $T$ be a traffic matrix and let $T^*$ be its symmetric counterpart. Then*

$$\min_{(x,y)\in U(T)} f(x, y) = \min_{(x,y)\in B(2T^*)} g(x, y),$$