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Bounds on Malapportionment

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ABSTRACT

Uniformly sized constituencies give voters similar influence on election outcomes. When constituencies are set up, seats are allocated to the administrative units, such as states or counties, using apportionment methods. According to the impossibility result of Balinski and Young, none of the methods satisfying basic monotonicity properties assign a rounded proportional number of seats (the Hare-quota). We study the malapportionment of constituencies and provide a simple bound as a function of the house size for an important class of divisor methods, a popular, monotonic family of techniques.

1. Introduction

In most democratic countries, some or all members of the Parliament are elected directly by the voters in electoral districts or (single-member) constituencies. For practical considerations these constituencies are embedded in the countries' existing administrative units, such as states or counties. To ensure equal representation, states are allotted seats in proportion to their populations. As fractional seats cannot be allocated, a fair division problem ensues. This is the so-called *apportionment problem*. Given an apportionment, the constituency boundaries can be designed in each region. This is also a non-trivial task as small towns cannot be split into two parts belonging to different constituencies. Thus, *districting* also makes proportional representation more difficult.

Proportional representation is not always pursued as a goal for all institutions (e.g. European Parliament, US Senate). Furthermore, some countries deliberately stray from proportional distribution to strengthen the representation of rural areas (e.g. Spain). Nevertheless, proportionality remains the fundamental principle of apportionment.

The 14th Amendment of the US Constitution already established that proportionality should be the key factor in apportionment. Since then, the US Supreme Court repeatedly confirmed that no deviation from equality is too small to challenge as long as a plan with less inequality can be presented (see the case Kirkpatrick v. Preisler (1969)). In Europe the Venice Commission, the advisory body of the Council of Europe in the field of constitutional law, published a guidebook for drafting electoral laws. The Code of Good Practice in Electoral Matters also attested that equality of voting power should be achieved by creating constituencies of equal size ([13], §13–15 in Section 2.2).

Even if the constituencies can be equalized within a state, there will be some deviations across states due to divisibility issues. The cited Supreme Court decision ordered the state of Missouri to redesign the districts because the attained 0.69% difference was not the lowest possible. In contrast, the constituencies of Montana are 88% larger than those of Rhode Island [2]. How much of this discrepancy is inherent? Is it possible to significantly decrease this gap? We aim to answer this question in this short note. We focus on apportionment, and disregard the difficulties that arise with the actual design of constituencies.

The Venice Commission itself advises that the gap should not be larger than 10% or, under exceptional circumstances, 15%. Since this requirement is hard to meet, many countries (including France, Germany, and Hungary) use a more relaxed interpretation: difference is measured from the average constituency size rather than pairwise. Indeed, the first draft of Hungary's redesigned electoral law in 2011 based on the stricter rule was mathematically impossible to satisfy. In the final version it was changed to the more relaxed interpretation.

What are feasible differences in general? We look at mainstream apportionment methods, establish bounds on the maximum of this difference as a function of the house size, and illustrate our results by data from Norway. Finally we note that the Impossibility Theorem of Balinski and Young [1] can often be resolved: certain methods, such as the Sainte-Laguë/Webster

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method, almost always satisfy the requirements, otherwise the Hare-quota requirement could be replaced by a softer condition as recommended by the Venice Commission.

2. Apportionment methods

We define the apportionment problem and methods. Let $N = \{1, 2, \ldots, n\}$ be the set of states of the country. An *apportionment problem* $(\boldsymbol{p}, H)$ is a pair consisting of a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ of state populations $p_i \in \mathbb{N}_+$ and a positive integer $H \in \mathbb{N}_+$ denoting the number of seats in the house. An *apportionment method* determines the non-negative integers $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = H$, specifying how many constituencies each of the states $1, 2, \ldots, n$ gets. Formally, it is a function M that assigns an allotment for each apportionment problem $(\boldsymbol{p}, H)$. Let $P = \sum_{i=1}^{n} p_i$ be the population of the country, and let $A = \frac{P}{H}$ denote the average size of a constituency. The fraction $\frac{p_i}{P} H = \frac{p_i}{A}$ is the *respective share* of state i.

Rounding the respective shares up or down is a natural way to obtain an apportionment. Apportionment methods that produce allotments by some form of rounding are said to exhibit the *Hare-quota*, or simply *quota* property. *Largest remainder methods* were explicitly designed with this property in mind. The most widely known method is the Hamilton-method, which first assigns the lower integer part of its respective share, the so-called *lower quota*, to each state, and then the remaining seats are distributed one-by-one to the states with the largest fractional parts of their respective shares.

Divisor methods constitute another family of apportionment techniques. An apportionment method is a *divisor method* if there exists a monotone increasing function $f : \mathbb{N} \to \mathbb{R}$, the *divisor criterion*, such that the seats are allocated to the state with the highest $\frac{p_i}{f(s)}$ value in each round. More precisely, suppose that $k-1$ seats are already allotted and the resulting apportionment is $\boldsymbol{a}$, then the kth seat goes to the state for which the fraction $\frac{p_i}{f(a_i)}$ is the highest. We assume that all of the $\frac{p_i}{f(a_i)}$ values are distinct. Ties are unlikely, for real data no tie-breaking rules are specified. The $\frac{p_i}{f(s)}$ value is the *rank-index* or *claim* of state i when it has s seats. Common divisor methods include the following (EP stands for Equal Proportions method — aliases are due to reinventions):

Adams method	$f(s) = s$
Danish method	$f(s) = s + 1/3$
Harmonic mean/Dean method	$f(s) = \dfrac{2s(s+1)}{2s+1}$
Huntington-Hill/EP method	$f(s) = \sqrt{s(s+1)}$
Sainte-Laguë/Webster method	$f(s) = s + 1/2$
Jefferson/D'Hondt method	$f(s) = s + 1$

The divisor criteria are listed in pointwise increasing order from Adams to Jefferson/D'Hondt; the methods favour large states over small states in the same order. That is, the Adams method favours small states the most, while the Jefferson/D'Hondt is the most beneficial for large states (see also Theorem 4 and [1,7,8]). The principal advantage of divisor methods is their immunity to paradoxes related to monotonicity, such as the Alabama-paradox.

We call divisor methods with $s \leq f(s) \leq s + 1$ *regular divisor methods*. More exotic divisor methods like the Imperiali ($f(s) = s + 2$) or the Macau ($f(s) = 2^s$) methods are not regular. Interestingly, while the Imperiali-method favours large states even more than the Jefferson/D'Hondt, the Macau-method is drastically small-state-friendly. Hence, it is false to conclude that the larger the divisor, the better it is for the large states.

The class of regular divisor methods is larger than it seems. The distribution of seats only depends on the relative order of claims, which does not change if all the claims are multiplied with the same (positive) number.

Remark 1. For any μ, ν such that $\frac{\nu}{\mu} \leq 1$ the divisor method with $f(s) = \mu s + \nu$ is regular and equivalent with the divisor method with $\hat{f}(s) = s + \nu/\mu$.

This explains, why the Sainte-Laguë/Webster method is sometimes defined with $f(s) = 2s + 1$.

A third branch of apportionment methods aims to minimize the range of populations. The *minimum range method* [3,4] minimizes the maximum disparity in representation between any two states, while the *Leximin* method [2], lexicographically minimizes the maximum departure, that is, the difference between the population of any constituency and the average constituency size.

Malapportionment measures have been studied by [6,10,11,14]. We look at departure from the average constituency size as a more explicit and intuitive measure of malapportionment.

3. Departure as a malapportionment measure

Let the relative difference displayed by the constituencies of state i be denoted by

$$\delta_i = \frac{\frac{p_i}{a_i} - A}{A},$$

and let $d_i = |\delta_i|$ be the *departure*, its absolute value. *Maximum departure* of an allotment, $\mathbf{a} = (a_1, \ldots, a_n)$ is the maximum of the d_i values for $i = 1, 2, \ldots, n$.

Let $l_i = \left\lfloor \frac{p_i}{P} H \right\rfloor$ and $u_i = \left\lceil \frac{p_i}{P} H \right\rceil$ denote the lower and upper quotas of state i, respectively, and let β_i (for *best* case) and ω_i (for *worst* case) denote the minimum and maximum differences achievable for state i when it gets the lower or upper integer part of its respective share.

$$\beta_i = \min\left(\left| \frac{\frac{p_i}{l_i} - A}{A} \right|, \left| \frac{\frac{p_i}{u_i} - A}{A} \right| \right), \qquad \beta = \max_{i \in N} \beta_i,$$

$$\omega_i = \max\left(\left| \frac{\frac{p_i}{l_i} - A}{A} \right|, \left| \frac{\frac{p_i}{u_i} - A}{A} \right| \right), \qquad \omega = \max_{i \in N} \omega_i.$$

Here β, the maximum of the β_i values, is a natural, not necessarily tight lower bound on the maximum departure for any apportionment. Similarly, the maximum of the ω_i values, denoted by ω, is an upper bound for any apportionment *which satisfies the Hare-quota*. If an apportionment does not satisfy Hare-quota, then it may have a departure larger than ω.

The β and ω bounds indicate that proportional representation relies on our ability to round the critical states in a good direction. Unfortunately, keeping the total at H forces us to allocate seats suboptimally. Suppose that there are seats left after an optimal rounding: Which state should we give them to? Should each state get only one extra seat (rounding it up rather than down as it would optimal)? Rounding in the wrong direction may increase departure drastically for small states, while for larger states even adding multiple seats has a minor effect on the relative difference, that is, departure, making such states ideal buffers to store seats that would mess up the apportionment. A similar argument applies to the case when the optimal allocation would distribute too many seats.

Enforcing quota ensures that the departure will not exceed ω, but the additional constraint also makes it difficult to stay close to β, since we cannot use states as buffers to lend/borrow problematic or desperately needed seats for critical states without creating too much inequality. What are these critical states? They are small states, which are only a few times the size of the average